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# NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

## TECHNICAL NOTE 2341

## A LEAST SQUARES CURVE FITTING METHOD WITH APPLICATIONS TO THE CALCULATION OF STABILITY COEFFICIENTS FROM TRANSIENT-RESPONSE DATA

By Marvin Shinbrot

Ames Aeronautical Laboratory
Moffett Field, Calif.



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# ERRATA

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| Page | Line | In Place of | Read |
|---|---|---|---|
| 4 | 2 | $\lambda_i \neq \lambda_j,\ i \neq j$ | $\lambda_i \neq \lambda_j$ if $i \neq j$ |
| 4 | 27 | $A_1$ | $A_j$ |
| 5 | 4 | equation (6) | equation (5) |
| 7 | 24 | m-1 | m+1 |
| 8 | 2 | m-1 | m+1 |
| 8 | 4 | m-1 | m+1 |
| 8 | 10 | equation (1) | equation (2) |
| 8 | 11 | $(D^2 - a_1 D - a_o)\ q(t) = (C_1 D - C_o)\ F(t)$ | $(D^2 + a_1 D + a_o)\ q(t) = (C_1 D + C_o)\ F(t)$ |
| 10 | 6 | (5a) | (4a) |
| 11 | 19 | $\sum_{i=0}^{\nu-1} \left[ \Delta q(t_i) - \Delta q_m(t_i) \right]$ | $\sum_{i=0}^{\nu-1} \left[ \Delta q(t_i) - \Delta q_m(t_i) \right]^2$ |
| 13 | 4 | $a_1 = \lambda_1 + \lambda_2 = 2l$ | $a_1 = -(\lambda_1 + \lambda_2) = -2l$ |
| 14 | 1 | $\beta = \frac{(l'\sigma - l\sigma')C_1 - \sigma' C_o + l' q_o}{2l\cdot}$ | $\beta = \frac{(l'\sigma - l\sigma')C_1 - \sigma' C_o + l' q_o}{2l'}$ |
| 14 | 16 | $(D^2 + bD + k)$ | $(D^2 + a_1 D + a_o)$ |
| 14 | 19 | q = . . . (4a) | q = . . . (4b) |
| 16 | 1<br>2 | $\int_o^T e^{-\lambda_i \tau} F(\tau) d\tau$ is constant and since $F(t)$ is constant for $t \geq T$, and $\int_T^t e^{-\lambda_i \tau} F(\tau) d\tau$ may be evaluated. | $\int_o^T e^{-\lambda_i \tau} F(\tau) d\tau$ is constant. Further, since $F(t)$ is constant for $t \geq T$, $\int_T^t e^{-\lambda_i \tau} F(\tau) d\tau$ may be evaluated. |
| 17 | 22 | $F_o$ | $F(0)$ |
| 19 | 24 | $\beta' \approx 2.6$ | $\beta' \approx -2.6$ |
| 22 | 6 | $\lambda l = l - l'i$ | $\lambda_2 = l - l'i$ |
| 24 | 15 | $2k\beta$ | $-2a_o\beta$ |
| 24 | 16 | $-\frac{2kl'\beta' + C_o l}{k}$ | $-\frac{2a_o l'\beta' + C_o l}{a_o}$ |
| 26 | 17<br>19 | $\int_o^t e^{\lambda_1 \tau} F(\tau) d\tau$ | $\int_o^t e^{-\lambda_1 \tau} F(\tau) d\tau$ |

| Page | Line | In Place of | Read |
|---|---|---|---|
| 26 | 17, 19 | $\int_0^t e^{\lambda_2 \tau} F(\tau)d\tau$ | $\int_0^t e^{-\lambda_2 \tau} F(\tau)d\tau$ |
| 28 | 7 | $\left(\frac{\partial q}{\partial \beta'}\right)_0 e^{-l_0 t}$ | $\frac{(\partial q/\partial \beta')_0}{e^{l_0 t}}$ |
| 30 | 1 | $\Sigma(\ddot{q} + b\dot{q} + kq - C_1\dot{F} - C_0F)^2$ | $\Sigma(\ddot{q} + a_1\dot{q} + a_0q - C_1\dot{F} - C_0F)^2$ |
| 31 | 24 | $\sum_{i=1}^{n} B_i e^{\lambda_i \Delta t}\left[\left(e^{\lambda_i \Delta t}\right)^n + Q_{n-1}\left(e^{\lambda_i \Delta t}\right)^{n-1} + \ldots + Q_0\right]$ | $\sum_{i=1}^{n} B_i e^{\lambda_i k \Delta t}\left[\left(e^{\lambda_i \Delta t}\right)^n + Q_{n-1}\left(e^{\lambda_i \Delta t}\right)^{n-1} + \ldots + Q_0\right]$ |
| 32 | 20 | $\sum_{k=0}^{\nu-1}\left(q_k - \sum_{i=1}^{n} B_i e^{\lambda_i t_k}\right)_1$ | $\sum_{k=0}^{\nu-1}\left(q_k - \sum_{i=1}^{n} B_i e^{\lambda_i t_k}\right)^2$ |
| 33 | 21 | equation (2) | equation (23) |
| 33 | 24 | $i=0, 1, 2, 3$ | $i=0, 1, 2$ |
| 33 | 26 | equation (4) | equation (25) |
| 37 | 3 | $k[\theta(t)-\theta(0-)]$ | $a_0[\theta(t)-\theta(0-)]$ |
| 37 | 3 | equation not numbered | equation (27) |
| 37 | 8 | $\int_{-\epsilon}^{t} \theta(t)dt$ | $\int_{-\epsilon}^{\epsilon} \theta(t)dt$ |
| 37 | 11 | equation (19) | equation (27) |
| 37 | 14 | equation (4a) | equation (4b) |
| 37 | 14 | $q(0+)$ | $\dot{q}(0+)$ |
| 37 | 15 | $q(t) = \frac{\lambda_1 q(0-) - \lambda_1\lambda_2 q(0) + C_1\lambda_1 + C_0}{\lambda_1(\lambda_1-\lambda_2)} e^{\lambda_1 t} + \frac{\lambda_2 q(0-) - \lambda_1\lambda_2 q(0) + C_1\lambda_2 + C_0}{\lambda_2(\lambda_2-\lambda_1)} e^{\lambda_2 t} + \frac{C_0}{\lambda_1\lambda_2}$ | $q(t) = \frac{\lambda_1 \dot{q}(0-) - \lambda_1\lambda_2 q(0) + C_1\lambda_1 + C_0}{\lambda_1(\lambda_1-\lambda_2)} e^{\lambda_1 t} + \frac{\lambda_2 \dot{q}(0-) - \lambda_1\lambda_2 q(0) + C_1\lambda_2 + C_0}{\lambda_2(\lambda_2-\lambda_1)} e^{\lambda_2 t} + \frac{C_0}{\lambda_1\lambda_2}$ |

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## SUMMARY

The problem of calculating airplane stability parameters from the aircraft response to an arbitrary disturbance is considered. To calculate the coefficients of the linear differential equation which describes the airplane transient response, application is made of a classical least-squares curve fitting method.

It is shown that the method is applicable, although somewhat cumbersome, when the input is an arbitrary function of time. Certain inputs are demonstrated to lead to simplification in the application of the method. Examples are given illustrating the means of using the method and showing its practicability.

Finally, an appendix, in which Prony's method and a generalization thereof appears, is presented.

## INTRODUCTION

The determination of the stability and control parameters of a dynamical system from its measured response has been a subject of increasing importance and interest, both from the standpoint of basic aerodynamic research and automatic stabilization of airplanes. This problem is essentially one of curve fitting and may be stated as follows: Given the time history of an airplane response to a known transient disturbance, and assuming a certain form of linear differential equation with constant coefficients describing the relation between response and disturbance, required to find the coefficients of this differential equation such that the sum of the squares of the differences between the given response and the one corresponding to the differential equation is a minimum.

A number of methods for the solution of this problem have been advanced by others and are noted in reference 1. However, these methods do not apply the least squares principle in the correct sense described above.

The method presented herein involves obtaining the solution of the differential equation in a form amenable to rigorous application of the method of least squares and such that a completely arbitrary input may be treated. A number of means are available for the fitting of a function in this form by least squares. The solution by means of a Taylor's series expansion seemingly gives good results and is used throughout this paper.

Considerable simplification of the method is possible when free oscillation data are available or when the input can be exactly or at least closely represented by a function the analytical form of which is known. Solutions for a pulse, a step, or a ramp input are given in detail or are indicated. The general solution for an arbitrary input is also included.

Selected examples of different inputs applied to the same physical system (airplane) are presented, which illustrate the different means of applying the method.

The report has been so organized that the engineer who is not interested in the derivation of the formulas used may read the first section of the report entitled "Statement of the Problem" and proceed from there directly to the section on examples.

## METHOD OF ANALYSIS

### Statement of the Problem

Consider a quantity $q_m(t)$ which has been measured at a set $\left\{t_i\right\}$, $i = 0, 1, \ldots, \nu-1$ of $\nu$ values of $t$. Suppose the initial conditions on $q_m$, obtained from the experimental data, are

$$\left.\begin{aligned} q_m(0) &= q_o \\ \left(\frac{dq_m}{dt}\right)_{t=0} &= \dot{q}_o \\ &\cdots\cdots \\ \left(\frac{d^{n-1} q_m}{dt^{n-1}}\right)_{t=0} &= q_o^{(n-1)} \end{aligned}\right\} \qquad (1)$$

where $n$ is determined below.

A physical interpretation of these quantities may be had by considering the data $q_m(t)$ as a time history of the pitching velocity of an airplane in response to an elevator deflection $F(t)$.

Consider next the differential equation

$$P_0(D)q(t) = P_1(D)F(t) \tag{2}$$

where $D$ is the operator $d/dt$, $P_0 = D^n + a_{n-1} D^{n-1} + \ldots + a_0$ and $P_1 = C_m D^m + C_{m-1} D^{m-1} + \ldots + C_0$ are two polynomials in $D$, and $F(t)$ is a known forcing function. Consider the set of all possible solutions of this differential equation obtained by varying the constants $a_i$ and $C_j$. Let $q_c(t)$ be that solution of equation (2) for which

$$M = \sum_{i=0}^{\nu-1} \left[q(t_i) - q_m(t_i)\right]^2 \tag{3}$$

is a minimum, and subject to the same initial conditions as $q_m$

$$\left.\begin{aligned} q_c(0) &= q_0 \\ \left(\frac{dq_c}{dt}\right)_{t=0} &= \dot{q}_0 \\ &\cdots\cdots \\ \left(\frac{d^{n-1}q_c}{dt^{n-1}}\right)_{t=0} &= q_0^{(n-1)} \end{aligned}\right\} \tag{1a}$$

It is then desired to find those values of the $a_i$ and the $C_j$ which correspond to $q_c$. These values will be unique, since to each set of these constants there corresponds one and only one solution of the differential equation if the initial conditions are determined.

In most practical problems, $m$ is less than $n$, and this assumption will be made throughout this paper. The extension of the method described herein to the case where $m$ is equal to or greater than $n$ should be clear.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the zeros of the polynomial $P_o(D)$. It is now assumed that $\lambda_i \neq \lambda_j$, $i \neq j$. The extension of the following solution to the case where not all the $\lambda_i$ are distinct can be found in any textbook on elementary differential equations and needs not be included here (reference 2). It is well known that if

$$P_o'(D) = \frac{d}{dD} P_o(D)$$

the solution of equation (2) is

$$q(t) = \sum_{j=1}^{n} \left\{ e^{\lambda_j t} \left[ A_j + \frac{P_1(\lambda_j)}{P_o'(\lambda_j)} \int_o^t e^{-\lambda_j \tau} F(\tau) d\tau \right] \right\} \tag{4}$$

where the constants $A_j$ are functions of the initial conditions $q(0)$, $\left(\frac{dq}{dt}\right)_{t=0}, \ldots, \left(\frac{d^{n-1} q}{dt^{n-1}}\right)_{t=0}, F(0), \left(\frac{dF}{dt}\right)_{t=0}, \ldots, \left(\frac{d^{m-1} F}{dt^{m-1}}\right)_{t=0}$ and of the constants $C_1, C_2, \ldots, C_m$. The method described in this paper consists of fitting $q(t)$, by a least squares procedure, to a function of the form (4). Several methods for fitting such a function may be found in the literature. In this report the classical device of linearization and iteration by means of a Taylor's series (reference 3, p. 214) is used. Therefore, a detailed study of this linearization follows, but it must be understood that it is in no way essential to the method. Fitting by steepest descents (reference 4), for example, could be used in place of the Taylor's series iteration. The mechanics of this iteration are cumbersome in the general case. They will, therefore, be described for particular cases, thus, it is hoped, making the method clear for any special case which may arise.

## Formulation of the Method When Free Oscillation Data Are Available

The method.– Suppose there exists a $T$ such that $F(t)=0$ for all $t \geq T$. Then, for $t \geq T$, the expression

$$\left[ A_1 + \frac{P_1(\lambda_j)}{P_o'(\lambda_j)} \int_o^t e^{-\lambda_j \tau} F(\tau) d\tau \right]$$

occurring in equation (4) is constant. Call this constant $B_j$.

Then for $t \geq T$

$$q(t) = \sum_{j=1}^{n} B_j e^{\lambda_j t} \tag{5}$$

By some means (see e.g., reference 1), first approximations to the constants occurring in equation (6) are found. Let $B_1^{(o)}$, $B_2^{(o)}$, . . . , $B_n^{(o)}$, $\lambda_1^{(o)}$, $\lambda_2^{(o)}$, . . . , $\lambda_n^{(o)}$ be the approximations to $B_1$, $B_2$, . . . , $B_n$, $\lambda_1$, $\lambda_2$, . . . , $\lambda_n$, respectively. The function $q(t)$ is now expanded by means of a Taylor series about the "point"

$$\left[B_1^{(o)}, \ . \ . \ . \ , B_n^{(o)}, \lambda_1^{(o)}, \ . \ . \ . \ , \lambda_n^{(o)}\right]$$

with all terms of order higher than the first omitted. From equation (5),

$$\frac{\partial q}{\partial B_i} = e^{\lambda_i t}, \ \frac{\partial q}{\partial \lambda_i} = B_i \ t \ e^{\lambda_i t}, \ i = 1, \ . \ . \ . \ , n$$

Denote the values of $q$, $\frac{\partial q}{\partial B_i}$, $\frac{\partial q}{\partial \lambda_i}$ at the "point" $\left[B_1^{(o)}, \ . \ . \ . \ , B_n^{(o)}, \lambda_1^{(o)}, \ . \ . \ . \ , \lambda_n^{(o)}\right]$ by $q^{(o)}$, $\left(\frac{\partial q}{\partial B_i}\right)^{(o)}$, $\left(\frac{\partial q}{\partial \lambda_i}\right)^{(o)}$, respectively. Then by Taylor's theorem, omitting all higher order terms,

$$\Delta q = \sum_{i=1}^{n} \left(\frac{\partial q}{\partial B_i}\right)^{(o)} \Delta B_i + \sum_{i=1}^{n} \left(\frac{\partial q}{\partial \lambda_i}\right)^{(o)} \Delta \lambda_i \tag{6}$$

where

$$\Delta q = q - q^{(o)}$$

$$\Delta B_i = B_i - B_i^{(o)}$$

$$\Delta \lambda_i = \lambda_i - \lambda_i^{(o)}$$

$$i=1, \ . \ . \ . \ , n$$

Therefore, from equation (6)

$$\Delta q = \sum_{i=1}^{n} e^{\lambda_i^{(o)} t} \left[ \Delta B_i + B_i^{(o)} t \, \Delta\lambda_i \right] \tag{7}$$

The desired equations which lead to the minimization of $M = \sum_{i=0}^{\nu-1} \left[ q_c(t_i) - q_m(t_i) \right]^2$ may now be found as follows: Let

$$\Delta q_m = q_m - q^{(o)}$$

Then

$$M = \sum_{i=0}^{\nu-1} \left[ \Delta q(t_i) - \Delta q_m(t_i) \right]^2$$

The equations for the increments then become

$$\left.\begin{aligned} \frac{\partial M}{\partial(\Delta B_i)} &= 0 \\ \frac{\partial M}{\partial(\Delta\lambda_i)} &= 0 \\ i &= 1, \ldots, n \end{aligned}\right\} \tag{8}$$

Since the expression (7) for $\Delta q$ is linear in $(\Delta B_i)$ and $(\Delta\lambda_i)$, the equations (8) will be precisely those found when fitting redundant data to a linear function by least squares (reference 3, pp. 209-211). Equations (8) are solved for the increments $\Delta B_i$ and $\Delta\lambda_i$, and the resulting values are added to $B_i^{(o)}$ and $\lambda_i^{(o)}$, respectively, to give the second approximation.

The iteration process by means of Taylor's expansions is now repeated until two successive iterations give the same values for the parameters to the desired number of significant figures.

It is of course true that in many cases, especially those in which very good data are available, no such Taylor series expansion is necessary, Prony's method (appendix A) giving sufficiently accurate values of the parameters. Examples have been found, however, where Prony's method did not lead to sufficiently good values, and the Taylor series was essential.

Final calculation of the coefficients of $P_o(D)$ and $P_1(D)$.— All that now remains for the problem to be completely solved is the computation of the coefficients of $P_o(D)$ and $P_1(D)$ from the calculated values of $B_i$ and $\lambda_i$.

Since $P_o(D) = D^n + a_{n-1} D^{n-1} + \ldots + a_o$, and since $\lambda_1, \ldots, \lambda_n$ are the zeros of this polynomial,

$$\left.\begin{aligned} a_{n-1} &= -\sum_{i=1}^{n} \lambda_i \\ a_{n-2} &= \sum_{\substack{i,j=1 \\ i<j}}^{n} \lambda_i \lambda_j \\ &\ldots\ldots \\ a_o &= (-1)^n \prod_{i=1}^{n} \lambda_i \end{aligned}\right\} \tag{9}$$

(reference 5, p. 29).

The constants $B_i$ were defined as follows:

$$B_i = A_i + \frac{P_1(\lambda_i)}{P_o'(\lambda_i)} \int_o^T e^{-\lambda_i \tau} F(\tau) d\tau$$

where $T$ is that value of $t$ where free oscillation begins. It is well known that the $A_i$ may be solved for as functions of the constants $C_1, \ldots, C_m$. The integrals occurring in the definition of $B_i$ can now be found graphically since $\lambda_i$ and $F(t)$ are known. There are, therefore, $n$ equations in the $m-1$ unknowns $C_o, \ldots, C_m$ (the

coefficients of $P_1$). It was assumed that m is less than n. Therefore, m-1 is not greater than n. If m-1 is equal to n, the number of equations is equal to the number of unknowns, and the constants $C_i$ may be determined. If m-1 is less than n, a least square procedure for solving redundant linear equations may again be used (reference 3, pp. 209-211).

## The Method for a Second-Order System

A case of great practical interest in aerodynamics is that of the second order, and in the present section this case will be treated in detail. A second-order system is one in which equation (1) becomes

$$(D^2 - a_1 D - a_o)\, q(t) = (C_1 D - C_o)\, F(t)$$

An example of such a second-order system which occurs in aerodynamics may be obtained by letting q be the response in pitching velocity to an elevator input F(t). In this case (reference 1),

$$a_1 = \frac{L_\alpha}{mV_o} - \frac{M_{\dot{\alpha}}}{I_y} - \frac{M_q}{I_y}$$

$$a_o = -\frac{L_\alpha}{mV_o} \times \frac{M_q}{I_y} - \frac{M_\alpha}{I_y}$$

$$C_o = \frac{M_\delta}{I_y} \times \frac{L_\alpha}{mV_o} - \frac{M_\alpha}{I_y} \times \frac{L_\delta}{mV_o}$$

$$C_1 = \frac{M_\delta}{I_y} - \frac{L_\delta}{mV_o} \times \frac{M_{\dot{\alpha}}}{I_y}$$

where

m mass of airplane

$V_o$ trim velocity of airplane

$I_y$ pitching moment of inertia

| | |
|---|---|
| $L$ | lift force |
| $M$ | pitching moment |
| $\delta$ | elevator input $F(t)$ |
| $\alpha$ | angle of attack |
| $\dot{\alpha}$ | $\frac{d\alpha}{dt}$ |
| $L_\delta$ | $\frac{\partial L}{\partial \delta}$ |
| $L_\alpha$ | $\frac{\partial L}{\partial \alpha}$ |
| $M_\delta$ | $\frac{\partial M}{\partial \delta}$ |
| $M_\alpha$ | $\frac{\partial M}{\partial \alpha}$ |
| $M_{\dot{\alpha}}$ | $\frac{\partial M}{\partial \dot{\alpha}}$ |
| $M_q$ | $\frac{\partial M}{\partial q}$ |

The general solution of $(D^2 + a_1 D + a_o)q = (C_1 D + C_o)F$ may be obtained from equation (4) which becomes, if $\lambda_1$ and $\lambda_2$ are the roots of $x^2 + a_1 x + a_o = 0$,

$$q = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + \frac{C_1\lambda_1 + C_o}{\lambda_1 - \lambda_2} e^{\lambda_1 t} \int_o^t e^{-\lambda_1 \tau} F(\tau) d\tau +$$

$$\frac{C_1\lambda_2 + C_o}{\lambda_2 - \lambda_1} e^{\lambda_2 t} \int_o^t e^{-\lambda_2 \tau} F(\tau) d\tau$$

Differentiating, an expression for $\dot{q} = \frac{dq}{dt}$ may be found, and letting $t=0$, it may be seen that

$$A_1 + A_2 = q(0)$$

$$A_1\lambda_1 + A_2\lambda_2 = \dot{q}(0) - C_1 F(0)$$

or, using equation (1a),

$$A_1 = \frac{\dot{q}_o - q_o\lambda_2 - C_1F_o}{\lambda_1 - \lambda_2}$$

$$A_2 = \frac{\dot{q}_o - q_o\lambda_1 - C_1F_o}{\lambda_2 - \lambda_1}$$

Therefore,

$$q = \frac{\dot{q}_o - q_o\lambda_2 - C_1F_o + (C_1\lambda_1 + C_o)\int_0^t e^{-\lambda_1\tau}\,F(\tau)d\tau}{\lambda_1 - \lambda_2}\,e^{\lambda_1 t} +$$

$$\frac{\dot{q}_o - q_o\lambda_1 - C_1F_o + (C_1\lambda_2 + C_o)\int_0^t e^{-\lambda_2\tau}\,F(\tau)d\tau}{\lambda_2 - \lambda_1}\,e^{\lambda_2 t} \qquad (5a)$$

Pulse input.- When a pulse-type input has been applied to a second-order system, that is, when there is a $T$ such that $F(t)=0$ for all $t \geq T$,

$$\left.\begin{aligned} B_1 &= A_1 + \frac{C_1\lambda_1 + C_o}{\lambda_1 - \lambda_2}\int_0^T e^{-\lambda_1\tau}\,F(\tau)d\tau \\ B_2 &= A_2 + \frac{C_1\lambda_2 + C_o}{\lambda_2 - \lambda_1}\int_0^T e^{-\lambda_2\tau}\,F(\tau)d\tau \end{aligned}\right\} \qquad (10)$$

using the same notation as before. Then, for $t \geq T$,

$$q = B_1\,e^{\lambda_1 t} + B_2\,e^{\lambda_2 t}$$

Suppose now, as is generally the case, that $\lambda_1$ and $\lambda_2$ are conjugate complex numbers. The form previously used for the description of the general method may still be followed in this case; however, simplifications occur if this notation is abandoned. If $\lambda_1$ and $\lambda_2$ are conjugate complex, $B_1$ and $B_2$ must be so also, since $q$ is real. Let

$$\left.\begin{aligned} \lambda_1 &= l + l'i \\ B_1 &= \beta + \beta' i \end{aligned}\right\} \quad (11)$$

Then

$$\lambda_2 = l - l'i, \; B_2 = \beta - \beta' i$$

and

$$q = 2e^{lt}\,(\beta \cos l't - \beta' \sin l't) \quad (12)$$

It is assumed that at this point, first approximations $l_o$, $l_o'$, $\beta_o$, $\beta_o'$ to the constants $l$, $l'$, $\beta$, $\beta'$, respectively, have been obtained. (See appendix A for one method for finding such approximations.)

The formulas needed for the Taylor's series iteration will now be derived. From equation (12),

$$\left.\begin{aligned} \frac{\partial q}{\partial l} &= 2te^{lt}\,(\beta \cos l't - \beta' \sin l't) \\ \frac{\partial q}{\partial l'} &= -\,2te^{lt}\,(\beta \sin l't + \beta' \cos l't) \\ \frac{\partial q}{\partial \beta} &= 2e^{lt} \cos l't \\ \frac{\partial q}{\partial \beta'} &= -\,2e^{lt} \sin l't \end{aligned}\right\} \quad (13)$$

Using a similar notation to that used in the previous section on the Taylor's series expansion,

$$M = \sum_{i=0}^{\nu-1} \left[\Delta q(t_i) - \Delta q_m(t_i)\right]$$

$$= \sum_{i=0}^{\nu-1} \left[\left(\frac{\partial q}{\partial l}\right)^{(o)} \Delta l + \left(\frac{\partial q}{\partial l'}\right)^{(o)} \Delta l' + \left(\frac{\partial q}{\partial \beta}\right)^{(o)} \Delta \beta + \left(\frac{\partial q}{\partial \beta'}\right)^{(o)} \Delta \beta' + q^{(o)} - q_m\right]^2$$

Taking the derivatives of $M$ with respect to $\Delta l$, $\Delta l'$, $\Delta \beta$, $\Delta \beta'$ and setting them equal to zero will result in the equations, the solution of which will give those values of the increments which minimize $M$. These equations thus become

$$\Delta l \sum \left[\left(\frac{\partial q}{\partial l}\right)^{(o)}\right]^2 + \Delta l' \sum \left(\frac{\partial q}{\partial l}\right)^{(o)} \left(\frac{\partial q}{\partial l'}\right)^{(o)} + \Delta\beta \sum \left(\frac{\partial q}{\partial l}\right)^{(o)} \left(\frac{\partial q}{\partial \beta}\right)^{(o)} +$$

$$\Delta\beta' \sum \left(\frac{\partial q}{\partial l}\right)^{(o)} \left(\frac{\partial q}{\partial \beta'}\right)^{(o)} = \sum \left[q_m - q^{(o)}\right] \left(\frac{\partial q}{\partial l}\right)^{(o)}$$

$$\Delta l \sum \left(\frac{\partial q}{\partial l'}\right)^{(o)} \left(\frac{\partial q}{\partial l}\right)^{(o)} + \Delta l' \sum \left[\left(\frac{\partial q}{\partial l'}\right)^{(o)}\right]^2 + \Delta\beta \sum \left(\frac{\partial q}{\partial l'}\right)^{(o)} \left(\frac{\partial q}{\partial \beta}\right)^{(o)} +$$

$$\Delta\beta' \sum \left(\frac{\partial q}{\partial l'}\right)^{(o)} \left(\frac{\partial q}{\partial \beta'}\right)^{(o)} = \sum \left[q_m - q^{(o)}\right] \left(\frac{\partial q}{\partial l'}\right)^{(o)}$$

$$\Delta l \sum \left(\frac{\partial q}{\partial \beta}\right)^{(o)} \left(\frac{\partial q}{\partial l}\right)^{(o)} + \Delta l' \sum \left(\frac{\partial q}{\partial \beta}\right)^{(o)} \left(\frac{\partial q}{\partial l'}\right)^{(o)} + \Delta\beta \sum \left[\left(\frac{\partial q}{\partial \beta}\right)^{(o)}\right]^2 +$$

$$\Delta\beta' \sum \left(\frac{\partial q}{\partial \beta}\right)^{(o)} \left(\frac{\partial q}{\partial \beta'}\right)^{(o)} = \sum \left[q_m - q^{(o)}\right] \left(\frac{\partial q}{\partial \beta}\right)^{(o)}$$

$$\Delta l \sum \left(\frac{\partial q}{\partial \beta'}\right)^{(o)} \left(\frac{\partial q}{\partial l}\right)^{(o)} + \Delta l' \sum \left(\frac{\partial q}{\partial \beta'}\right)^{(o)} \left(\frac{\partial q}{\partial l'}\right)^{(o)} +$$

$$\Delta\beta \sum \left(\frac{\partial q}{\partial \beta'}\right)^{(o)} \left(\frac{\partial q}{\partial \beta}\right)^{(o)} + \Delta\beta' \sum \left[\left(\frac{\partial q}{\partial \beta'}\right)^{(o)}\right]^2$$

$$= \sum \left[q_m - q^{(o)}\right] \left(\frac{\partial q}{\partial \beta'}\right)^{(o)} \tag{14}$$

As before, it is now assumed that this iteration is repeated until two successive iterations give the same values for the parameters to the desired number of significant figures.

The parameters $l$, $l'$, $\beta$ and $\beta'$ have thus been determined. The solution to our problem will be complete if we can, from these values, calculate $a_o$, $a_1$, $C_o$, $C_1$. From equations (9),

$$a_1 = \lambda_1 + \lambda_2 = 2l$$

$$a_2 = \lambda_1\lambda_2 = l^2 + l'^2$$

A means of finding $C_1$ and $C_o$ must now be found. In order to accomplish this, consider

$$\int_0^T e^{-\lambda_1\tau} F(\tau)d\tau = \int_0^T e^{-l\tau} \cos l'\tau\, F(\tau)d\tau$$

$$-i \int_0^T e^{-l\tau} \sin l'\tau\, F(\tau)d\tau$$

Let

$$\sigma = \int_0^T e^{-l\tau} \cos l'\tau\, F(\tau)d\tau$$

Let

$$\sigma' = \int_0^T e^{-l\tau} \sin l'\tau\, F(\tau)d\tau$$

Then

$$\int_0^T e^{-\lambda_1\tau} F(\tau)d\tau = \sigma - \sigma' i$$

and since $\lambda_2$ is the conjugate of $\lambda_1$,

$$\int_0^T e^{-\lambda_2\tau} F(\tau)d\tau = \sigma + \sigma' i$$

Recalling that $B_1 = \beta + \beta' i$, it follows from equation (4a) that

$$\left.\begin{aligned}\beta &= \frac{(l'\sigma - l\sigma')C_1 - \sigma' C_o + l' q_o}{2l'} \\ \beta' &= \frac{l q_o - \dot{q}_o - [l\sigma + l'\sigma' - F(0)]C_1 - \sigma C_o}{2l'}\end{aligned}\right\} \qquad (15)$$

These equations may be solved for $C_o$ and $C_1$.

<u>The method when the forcing function continues throughout the motion</u>.— It is now supposed that in the given data, there is no $T$ such that the forcing function $F(t)=0$ for all $t \geq T$. In this case, the coefficient of $e^{\lambda_j t}$ in the expression for $q(t)$ given by equation (4) is no longer constant. Prony's method cannot, therefore, be used for the first approximation. The simplification which occurs in the Taylor's expansion in the previous case due to the fact that $q(t)$ is a sum of exponentials with constant coefficients is also not present here.

There are cases, however, even when free oscillation does not occur where the present method can be applied with little difficulty. The best examples of such cases are those in which $F(t)$ is known to have a certain analytical form. As an example, the two degrees of freedom system $(D^2 + bD + k)\, q(t) = (C_1 D + C_o)\, F(t)$ may be considered, where

$F(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$, the so-called "step" function.

For this input, equation (4a) becomes

$$q = \frac{\lambda_1 \dot{q}_o - \lambda_1\lambda_2 q_o + C_o}{\lambda_1(\lambda_1 - \lambda_2)} e^{\lambda_1 t} + \frac{\lambda_2 \dot{q}_o - \lambda_1\lambda_2 q_o + C_o}{\lambda_2(\lambda_2 - \lambda_1)} e^{\lambda_2 t} + \frac{C_o}{\lambda_1\lambda_2} \qquad (4a)$$

where, since $dq/dt$ is discontinuous at $t=0$ if $F(t)$ is a step, $\dot{q}_o = \lim\limits_{t \to 0+} \frac{dq}{dt}$, $t \to 0+$ indicating that $t$ is to approach zero through positive values only. The apparent anomaly that $q$ is independent of $C_1$ is resolved in appendix B, where it is shown that

$$\dot{q}(0+) = \dot{q}(0-) + C_1 F(0+) = \dot{q}(0-) + C_1$$

First approximations to the constants

$$B_1 = \frac{\lambda_1\dot{q}(0+)-\lambda_1\lambda_2 q_o+C_o}{\lambda_1(\lambda_1-\lambda_2)} = \frac{\lambda_1\dot{q}(0-)-\lambda_1\lambda_2 q_o+C_1\lambda_1+C_o}{\lambda_1(\lambda_1-\lambda_2)}$$

$$B_2 = \frac{\lambda_2\dot{q}(0+)-\lambda_1\lambda_2 q_o+C_o}{\lambda_2(\lambda_2-\lambda_1)} = \frac{\lambda_2\dot{q}(0-)-\lambda_1\lambda_2 q_o+C_1\lambda_2+C_o}{\lambda_2(\lambda_2-\lambda_1)}$$

$$B_3 = \frac{C_o}{\lambda_1\lambda_2}$$

which occur in equation (4b) can be found by a simple extension to Prony's method. (See appendix A.) A Taylor's expansion may then be applied to obtain a closer approximation.

A very important input, closely related to the step is the so-called "pseudo-step." In any physical case, an exact step can never be obtained for an input. The input will always have a certain finite slope near $t=0$; there may also be a certain amount of overshoot or, reciprocally, the input may undershoot its steady state value. Any input which rapidly (but not instantaneously) attains a constant nonzero value will be called a pseudo-step. Suppose then that $F(t)$ is such a function. Then there is a value $T$ such that $F(t)$ is a constant for all $t \geq T$. Suppose this constant is $c$. Let this function be applied to the second-order system $(D^2 + a_1 + a_o)\ q(t) = (C_1 D + C_o)\ F(t)$. The response in $q$ is then given by equation (4a). It is here assumed for simplicity that the initial conditions $q(0)$ and $\dot{q}(0)$ are both zero; $F(0)$ is also zero. Then

$$q(t) = \frac{C_1\lambda_1+C_o}{\lambda_1-\lambda_2}\, e^{\lambda_1 t}\int_0^t e^{-\lambda_1\tau}\, F(\tau)d\tau + \frac{C_1\lambda_2+C_o}{\lambda_2-\lambda_1}\, e^{\lambda_2 t}\int_0^t e^{-\lambda_2\tau}\, F(\tau)d\tau$$

For $t \geq T$,

$$q(t) = \frac{C_1\lambda_1+C_o}{\lambda_1-\lambda_2}\, e^{\lambda_1 t}\left[\int_0^T e^{-\lambda_1\tau}\, F(\tau)d\tau + \int_T^t e^{-\lambda_1\tau}\, F(\tau)d\tau\right] +$$

$$\frac{C_1\lambda_2+C_o}{\lambda_2-\lambda_1}\, e^{\lambda_2 t}\left[\int_0^T e^{-\lambda_2\tau}\, F(\tau)d\tau + \int_T^t e^{-\lambda_2\tau}\, F(\tau)d\tau\right]$$

$\int_0^T e^{-\lambda_1\tau} F(\tau)d\tau$ is constant and since $F(t)$ is constant for $t \geq T$, and $\int_T^t e^{-\lambda_1\tau} F(\tau)d\tau$ may be evaluated.

Let

$$\frac{c}{\lambda_1}\,\frac{C_1\lambda_1+C_0}{\lambda_1-\lambda_2} = G_1, \quad \frac{c}{\lambda_2}\,\frac{C_1\lambda_2+C_0}{\lambda_2-\lambda_1} = G_2,$$

$$\frac{C_1\lambda_1+C_0}{\lambda_1-\lambda_2}\int_0^T e^{-\lambda_1\tau} F(\tau)d\tau = H_1, \quad \frac{C_1\lambda_2+C_0}{\lambda_2-\lambda_1}\int_0^T e^{-\lambda_2\tau} F(\tau)d\tau = H_2$$

Then $G_1$, $G_2$, $H_1$, $H_2$ are constants, and

$$\begin{aligned} q(t) &= H_1 e^{\lambda_1 t} + G_1\left[e^{\lambda_1(t-T)}-1\right] + H_2 e^{\lambda_2 t} + G_2\left[e^{\lambda_2(t-T)}-1\right] \\ &= \left(H_1 + \frac{G_1}{e^{\lambda_1 T}}\right) e^{\lambda_1 t} + \left(H_2 + \frac{G_2}{e^{\lambda_2 T}}\right) e^{\lambda_2 t} - (G_1 + G_2) \end{aligned}$$

Thus $q(t)$ is the sum of two exponentials plus a constant. First approximations may now be found by the extension to Prony's method given in appendix A, and Taylor's series expansions may be used to improve these values.

Another simple example which may be considered is that where $F(t) = \begin{cases} 0, & t < 0 \\ t, & t \geq 0 \end{cases}$. In this case the response of the second-order system would be

$$q = \gamma_1 e^{\lambda_1 t} + \gamma_2 e^{\lambda_2 t} - (\gamma_1\lambda_1+\gamma_2\lambda_2)\,t - (\gamma_1+\gamma_2)$$

if

$$q(0) = \dot{q}(0) = 0$$

where

$$\gamma_1 = \frac{C_1\lambda_1+C_0}{\lambda_1^2(\lambda_1-\lambda_2)}, \quad \gamma_2 = \frac{C_1\lambda_2+C_0}{\lambda_2^2(\lambda_2-\lambda_1)}$$

A first approximation may be found by several means, the derivative method (see example 3) being only one of these. Again, Taylor's series may be applied to improve these values.

In the most general case, where $F(t)$ is only known graphically or tabularly, the method can still be applied; there is a certain amount of added labor involved, but in case such an example should occur, the method for this general case is outlined below. It is to be noted that this general case is better suited to computation with a high-speed electronic computer than it is to manual computation. However, it may certainly be applied manually.

The entire method proceeds from equation (4). It is assumed that the given differential equation has been solved for $q$ in this form.

Step 1.- By some means, a first approximation to the parameters must be found (several such methods may be found in reference 1).

Step 2.- The function $q(t)$ is expanded in a Taylor's series, all teams of order higher than the first being omitted. Herein lies most of the computation, since $\int_0^t e^{-\lambda_1 \tau} F(\tau)d\tau$ must be found graphically as a function of $t$.

As an example, consider again the second-order case $(D^2 + a_1D + a_o)$ $q(t) = (C_1D + C_o)\ F(t)$, where $F(t)$ is given graphically. It is assumed for simplicity that

$$F_o = q(0) = \dot{q}(0) = 0$$

Then, using the same notation as before, $A_1 = A_2 = 0$, and $q(t)$ is simply the particular integral:

$$q(t) = \frac{C_1\lambda_1+C_o}{\lambda_1-\lambda_2}\ e^{\lambda_1 t} \int_0^t e^{-\lambda_1 \tau}\ F(\tau)d\tau + \frac{C_1\lambda_2+C_o}{\lambda_2-\lambda_1}\ e^{\lambda_2 t} \int_0^t e^{-\lambda_2 \tau}\ F(\tau)d\tau$$

The computations (especially when it comes to taking derivatives for use in the Taylor series) are greatly simplified by the substitution $B_1 = \dfrac{C_1\lambda_1+C_o}{\lambda_1-\lambda_2}$ $B_2 = \dfrac{C_1\lambda_2+C_o}{\lambda_2-\lambda_1}$. The problem is then restated as follows: To find the best fit for $q(t)$ to a function of the form

$$q(t) = B_1\ e^{\lambda_1 t}\int_0^t e^{-\lambda_1 \tau}\ F(\tau)d\tau + B_2\ e^{\lambda_2 t} \int_0^t e^{-\lambda_2 \tau}\ F(\tau)d\tau$$

Minimization of $M = \Sigma_1(q_c - q_m)^2$ with respect to $\beta$, $\beta'$, $l$, $l'$, (where $\beta + \beta' i = B_1$, $\beta - \beta' i = B_2$, $l + l' i = \lambda_1$, $l - l' i = \lambda_2$) is equivalent to minimization with respect to $C_1$, $C_0$, $a_1$, and $a_0$, which is the desired minimization. The method now proceeds as before, with the first approximations being found and with the application of a Taylor's series.

Another means which might be employed in this general case would be to first fit $F(t)$ to some suitable function, and then find $q(t)$ as an explicit function of $t$ as was done above for the step and the ramp.

## EXAMPLES

The engineer who is interested primarily in applications may read these examples immediately after reading the first section of the report entitled "Statement of the Problem."

### Example I – Pulse Input

Consider the input shown in figure 1(a) applied to the second-order system

$$(D^2 + a_1 D + a_0)q(t) = (C_1 D + C_0)F(t) \tag{16}$$

The response is given in figure 1(b). The methods and formulas described previously can be used directly for this example. If $t \geq 0.4$,

$$q = B_1 e^{\lambda_1 t} + B_2 e^{\lambda_2 t} \tag{5a}$$

or, if $B_1 = \beta + \beta' i$, $\lambda_1 = l + l' i$, and $t \geq 0.4$,

$$\frac{q}{2} = e^{lt} (\beta \cos l' t - \beta' \sin l' t) \tag{12}$$

It is now assumed that by some means first approximation $l_0$, $l_0'$, $\beta_0$, $\beta_0'$, to $l$, $l'$, $\beta$, $\beta'$, respectively, have been found.[1] Prony's method (see appendix A) gives $l_0 = -0.91$, $l_0' = 7.02$, $\beta_0 = 0.3681$, $\beta_0' = -2.6775$.

From equation (12)

$$\left.\begin{aligned}
\frac{\partial}{\partial l}\left(\frac{q}{2}\right) &= t\, e^{lt} (\beta \cos l't - \beta' \sin l't) \\
\frac{\partial}{\partial l'}\left(\frac{q}{2}\right) &= -t\, e^{lt} (\beta \sin l't + \beta' \cos l't) \\
\frac{\partial}{\partial \beta}\left(\frac{q}{2}\right) &= e^{lt} \cos l't \\
\frac{\partial}{\partial \beta'}\left(\frac{q}{2}\right) &= -e^{lt} \sin l't
\end{aligned}\right\} \tag{13}$$

[1]Prony's method, for example, may be used here. Prony's method is actually applied to this example in appendix A. An ordinary period and damping analysis (described below), which is shorter than Prony's method, may also be used. The afore-mentioned period and damping analysis proceeds as follows: The minima of $q$ lie on a curve $q = K\, e^{lt}$, $K$ = constant. Let $q_1$ and $q_2$ be two successive minima of $q$. Then $\left(\frac{q_2}{q_1}\right) = e^{l\tau}$, $\tau$ = period of $q$ (which may be found from figure 1(b)). Therefore, $l = \frac{1}{\tau} \ln\left(\frac{q_2}{q_1}\right)$. As for $l'$, $l'$ is equal to $\frac{2\pi}{\tau}$. Then

$$q = 2\, e^{lt} (\beta \cos l't - \beta' \sin l't)$$

Fitting at any two points $t_1$ and $t_2$

$$(\cos l't_1)\, \beta - (\sin l't_1)\, \beta' = \frac{1}{2}\, e^{-lt_1}\, q(t_1)$$

$$(\cos l't_2)\, \beta - (\sin l't_2)\, \beta' = \frac{1}{2}\, e^{-lt_2}\, q(t_2)$$

which may be solved for $\beta$ and $\beta'$. This analysis gives $l \approx -0.9$, $l' \approx 7.0$. Letting $t_1 = 1.0$, $t_2 = 2.0$, it may be found that $\beta \approx 0.4$, $\beta' \approx 2.6$.

Therefore, by Taylor's theorem, omitting all higher order terms,

$$t(\beta \cos l't - \beta' \sin l't)\,\Delta l - t\,(\beta \sin l't + \beta' \cos l't)\,\Delta l' +$$

$$(\cos l't)\,\Delta\beta - (\sin l't)\,\Delta\beta' = \frac{q_m}{2} e^{-lt} - (\beta \cos l't - \beta' \sin l't)$$

Referring to table I where circled numbers refer to columns,

$$(17)\,\Delta l - (18)\,\Delta l' + (9)\,\Delta\beta - (10)\,\Delta\beta' = (19)$$

Minimizing M leads to the equations (see equations (14))

$$\Delta l\,\Sigma (17)^2 - \Delta l'\,\Sigma (17)(18) + \Delta\beta\,\Sigma (9)(17) - \Delta\beta'\,\Sigma (10)(17) = \Sigma (17)(19)$$

$$-\Delta l\,\Sigma (17)(18) + \Delta l'\,\Sigma (18)^2 - \Delta\beta\,\Sigma (9)(18) + \Delta\beta'\,\Sigma (10)(18) = -\Sigma (18)(19)$$

$$\Delta l\,\Sigma (9)(17) - \Delta l'\,\Sigma (9)(18) + \Delta\beta\,\Sigma (9)^2 - \Delta\beta'\,\Sigma (9)(10) = \Sigma (9)(19)$$

$$-\Delta l\,\Sigma (10)(17) + \Delta l'\,\Sigma (10)(18) - \Delta\beta\,\Sigma (9)(10) + \Delta\beta'\,\Sigma (10)^2 = -\Sigma (10)(19)$$

These equations give

$$\Delta l = -0.0100 \qquad \Delta l' = 0.0052$$

$$\Delta\beta = -0.0118 \qquad \Delta\beta' = -0.0320$$

or

$$l = -0.9200 \qquad l' = 7.0252$$

$$\beta = 0.3563 \qquad \beta' = -2.7095$$

A second iteration was now applied, and it was found that the increments were zero to four decimals.

Finally, for computing $a_1$, $a_0$, $C_1$ and $C_0$

$$a_1 = -2l = 1.84$$

$$a_0 = l^2 + l'^2 = 50.1998$$

The functions $e^{-lt} F(t) \cos l't$ and $e^{-lt} F(t) \sin l't$ are now tabulated and plotted (table II and fig. 2). These two functions must now be integrated to find $\sigma$ and $\sigma'$. It is to be noted that in this example $F(t)$ is exactly equal to $t$ for $0 \leq t \leq 0.2$, and to $(0.4 - t)$ for $0.2 \leq t \leq 0.4$, and so the integration may be done analytically. The integration may be done graphically by means of a planimeter, however, when $F(t)$ is not so clearly a perfect triangular pulse, as is usually the case. A planimeter was actually used in this example to give $\sigma = 0.00493$ and $\sigma' = 0.0405$. Equations (15) are now set up:

$$0.280\, C_1 + 0.00493\, C_0 = 38.0690$$

$$0.07189\, C_1 - 0.0405\, C_0 = 5.0061$$

which give $C_1 = 134.0$ and $C_0 = 114.2$, to four significant figures.

The true values of the parameters in the preceding example were

$$a_1 = 1.84$$

$$a_0 = 50.2$$

$$C_1 = 134.0$$

$$C_0 = 114.4$$

Example II – Step Input

Consider a "step" input $F(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$ applied to the system described by equation (16). The response is given by equation (4b).

From the response shown in figure 3 it may be seen that $q(0) = 0$, $\dot{q}(0-) = 0$. Then equation (4b) becomes

$$q(t) = \frac{C_1\lambda_1+C_o}{\lambda_1(\lambda_1-\lambda_2)} e^{\lambda_1 t} + \frac{C_1\lambda_2+C_o}{\lambda_2(\lambda_2-\lambda_1)} e^{\lambda_2 t} + \frac{C_o}{\lambda_1\lambda_2}$$

Let

$$B_1 = \frac{C_1\lambda_1+C_o}{\lambda_1(\lambda_1-\lambda_2)}$$

$$B_2 = \frac{C_1\lambda_2+C_o}{\lambda_2(\lambda_2-\lambda_1)}$$

$$B_3 = \frac{C_o}{\lambda_1\lambda_2}$$

Then, using the same notation as before, with $\lambda_1 = l + l'i$, $\lambda l = l - l'i$, $\lambda_1 + \lambda_2 = -a_1$, $\lambda_1\lambda_2 = a_o$,

$$B_1 = -\frac{C_o}{2a_o} - \frac{C_1a_o+C_ol}{2l'a_o} i$$

$$B_2 = -\frac{C_o}{2a_o} + \frac{C_1a_o+C_ol}{2l'a_o} i$$

Let $B_1 = \beta + \beta'i$. Then $\beta = -\dfrac{C_o}{2a_o}$, $\beta' = -\dfrac{C_1a_o+C_ol}{2l'a_o}$. But $B_3 = \dfrac{C_o}{a_o}$. Therefore, $B_3 = -2\beta$. Then $\dfrac{q}{2} = e^{lt}(\beta \cos l't - \beta' \sin l't) - \beta$.

First approximations may now be found by the extension to Prony's method described in appendix A or by a generalization of the ordinary period and damping analysis. For simplicity, this period and damping method was used to give $l_o = -0.91$, $l_o' = 7.0$, $\beta_o = -1.11$, $\beta_o' = -9.7$.

The Taylor's expansion is now applied

$$\frac{q}{2} = e^{lt} \, (\beta \cos l't - \beta' \sin l't) - \beta$$

$$\frac{\partial}{\partial l}\left(\frac{q}{2}\right) = t\, e^{lt} \, (\beta \cos l't - \beta' \sin l't)$$

$$\frac{\partial}{\partial l'}\left(\frac{q}{2}\right) = -\, t\, e^{lt} \, (\beta \sin l't + \beta' \cos l't)$$

$$\frac{\partial}{\partial \beta}\left(\frac{q}{2}\right) = e^{lt} \cos l't - 1$$

$$\frac{\partial}{\partial \beta'}\left(\frac{q}{2}\right) = -\, e^{lt} \sin l't$$

Therefore,

$$\frac{q}{2} = \left[e^{lt}(\beta \cos l't - \beta' \sin l't) - \beta\right] + \left[t\, e^{lt} \, (\beta \cos l't - \beta' \sin l't)\right] \Delta l - \left[t\, e^{lt} \, (\beta \sin l't + \beta' \cos l't)\right]\Delta l' + \left[e^{lt} \cos l't - 1\right]\Delta\beta - \left[e^{lt} \sin l't\right]\Delta\beta'$$

Or, referring to table III, (18) $\Delta l$ + (19) $\Delta l'$ + (15) $\Delta\beta$ − (10) $\Delta\beta'$ = (21).
Again minimizing $M$, it may be found that

$$\Delta l = -\, 0.0100$$

$$\Delta l' = 0.0242$$

$$\Delta\beta = -\, 0.0292$$

$$\Delta\beta' = 0.3110$$

which in turn give

$$l = -\ 0.9200$$

$$l' = 7.0242$$

$$\beta = -\ 1.1392$$

$$\beta' = -\ 9.3890$$

A second Taylor's expansion was then applied (see table IV), giving

$$l = -\ 0.9201$$

$$l' = 7.0258$$

$$\beta = -\ 1.1400$$

$$\beta' = -\ 9.3875$$

Another iteration would lead to increments which are zero to four decimals.

For the final calculation of the coefficients,

$$a_1 = -\ (\lambda_1+\lambda_2) = -\ 2l = 1.8402$$

$$a_0 = \lambda_1\lambda_2 = l^2 l'^2 = 50.2084$$

$$C_0 = 2k\beta = 114.4715$$

$$C_1 = -\frac{2kl'\beta'+C_0 l}{k} = 134.0072$$

It is to be noted that a second iteration such as was applied above was hardly important, since the results of the first Taylor's series give

$$a_1 = 1.84$$

$$a_o = 50.19$$

$$C_o = 114.34$$

$$C_1 = 134.00$$

the true values being, as before,

$$a_1 = 1.84$$

$$a_o = 50.2$$

$$C_o = 114.4$$

$$C_1 = 134.0$$

Example III – Input Requiring the General Method

Consider finally the input shown in figure 4(a) applied to the system

$$(D^2 + a_1 D + a_o)q(t) = (C_1 D + C_o)F(t) \tag{16}$$

Such an input might occur, for example, in a stabilized airplane where the pitching velocity $q$ is fed back to the elevator to change the input.

First approximations by the derivative method.– The so-called "derivative method" will here be applied in order to find first approximations to the desired constants. From figure 4, $\dot{F}$ and $\dot{q}$ are found graphically as functions of $t$. Then $\dot{q}$ is plotted (fig. 5), and from this, $\ddot{q}$ is found and tabulated. (See table V.) Rewriting equation (16),

$$(\dot{q})a_1 + (q)a_o - (\dot{F})C_1 - (F)C_o = -(\ddot{q})$$

and referring to table V,

$$⑥a_1 + ⑤a_0 - ④C_1 - ③C_0 = -⑦$$

A least-squares analysis is now applied to this equation, giving

$$a_1 = 1.84$$

$$a_0 = 50.19$$

$$C_1 = 133.89$$

$$C_0 = 114.91$$

The high accuracy of these first approximations is due, of course, to the excellence of the data in the example. The leading objection to the derivative method obviously is the necessity of finding the derivatives graphically, which in many cases leads to errors so gross as to make the values of the parameters found in this manner entirely valueless.

<u>The Taylor's series iteration.</u>— Since $F(t)$ is continuous at $t=0$, the constants $q(0)$, $\dot{q}(0)$, $F(0)$ of equation (4a) may be found as usual, by inspection of figures 4 and 5. Since $q(0) = \dot{q}(0) = F(0) = 0$, from equation (4a),

$$q = \frac{C_1\lambda_1+C_0}{\lambda_1-\lambda_2} e^{\lambda_1 t} \int_0^t e^{\lambda_1\tau} F(\tau)d\tau + \frac{C_1\lambda_2+C_0}{\lambda_2-\lambda_1} e^{\lambda_2 t} \int_0^t e^{\lambda_2\tau} F(\tau)d\tau$$

or, letting $B_1 = \frac{C_1\lambda_1+C_0}{\lambda_1-\lambda_2}$, $B_2 = \frac{C_1\lambda_2+C_0}{\lambda_2-\lambda_1}$,

$$q = B_1 e^{\lambda_1 t} \int_0^t e^{\lambda_1\tau} F(\tau)d\tau + B_2 e^{\lambda_2 t} \int_0^t e^{\lambda_2\tau} F(\tau)d\tau$$

Using the same notation as before, with $\lambda_1 = l + l'i$, $B_1 = \beta + \beta' i$, $\int_0^t e^{-\lambda_1\tau} F(\tau)d\tau = \sigma - \sigma' i$, it follows that $\lambda_2 = l - l'i$, $B_2 = \beta - \beta' i$, $\int_0^t e^{-\lambda_2\tau} F(\tau)d\tau = \sigma + \sigma' i$, and that,

$$q = 2\, e^{lt}[(\beta\sigma + \beta'\sigma') \cos l't + (\beta\sigma' - \beta'\sigma) \sin l't] \tag{17}$$

It will also be noted that $\beta = \frac{C_1}{2}$, $\beta' = -\frac{C_1 l + C_0}{2l'}$. Letting zero subscripts denote the values of the indicated parameters at the first approximations,

$$l_o = -\,0.92$$

$$l'_o = 7.02$$

$$\beta_o = 66.945$$

$$\beta'_o = -\,0.59$$

From equation (17)

$$\left.\begin{aligned}
\frac{\partial q}{\partial l} &= 2\, e^{lt}\Big\{[(\beta\sigma+\beta'\sigma')t - (\beta\rho+\beta'\rho')] \cos l't + \\
&\qquad [(\beta\sigma'-\beta'\sigma)t + (\beta'\rho-\beta\rho')] \sin l't\Big\} \\
\frac{\partial q}{\partial l'} &= 2\, e^{lt}\Big\{[(\beta\sigma'-\beta'\sigma)t + (\beta'\rho-\beta\rho')] \cos l't - \\
&\qquad [(\beta\sigma+\beta'\sigma')t - (\beta\rho+\beta'\rho')] \sin l't\Big\} \\
\frac{\partial q}{\partial \beta} &= 2\, e^{lt}\, (\sigma \cos l't + \sigma' \sin l't) \\
\frac{\partial q}{\partial \beta'} &= 2\, e^{lt}\, (\sigma' \cos l't - \sigma \sin l't)
\end{aligned}\right\} \tag{18}$$

where

$$\left\{\begin{aligned}
\rho &= \int_0^t t\, F(t)\, e^{-lt} \cos l't\, dt \\
\rho' &= \int_0^t t\, F(t)\, e^{-lt} \sin l't\, dt
\end{aligned}\right.$$

The function $F(t)$ is now multiplied by the four quantities $e^{-l_o t} \cos l'_o t$, $e^{-l_o t} \sin l'_o t$, $t\, e^{-l_o t} \cos l'_o t$, $t\, e^{-l_o t} \sin l'_o t$, point by point, and the resulting products are plotted against $t$ (fig. 6). These four curves are integrated as functions of $t$, giving, respectively, $\sigma$, $\sigma'$, $\rho$, and $\rho'$ (table VI).

The quantities $\dfrac{q_o}{e^{l_o t}}$, $\dfrac{(\partial q/\partial l)_o}{e^{l_o t}}$, $\dfrac{(\partial q/\partial l')_o}{e^{l_o t}}$, $\dfrac{(\partial q/\partial \beta)_o}{e^{l_o t}}$, and $\left(\dfrac{\partial q}{\partial \beta'}\right)_o e^{l_o t}$ are then computed from equation (18).

Then, using a Taylor's series and table VI,

$$(8)\Delta l + (9)\Delta l' + (10)\Delta\beta + (11)\Delta\beta' = (15)$$

Minimizing $M$ with respect to $\Delta l$, $\Delta l'$, $\Delta\beta$, $\Delta\beta'$ gives

$$\Delta l = -0.00$$

$$\Delta l' = 0.01$$

$$\Delta\beta = 0.17$$

$$\Delta\beta' = -0.05$$

Due to the inherent inaccuracy of a planimeter, only two decimal places were preserved here.

Thus

$$l = -0.92$$

$$l' = 7.03$$

$$\beta = 67.03$$

$$\beta' = -0.64$$

The desired parameters were finally computed to be:

$$a_1 = 1.84$$

$$a_o = 50.28$$

$$C_1 = 134.06$$

$$C_o = 114.69$$

## CONCLUDING REMARKS

A method has been described by which the coefficients of the differential equation

$$\frac{d^n q}{dt^n} + a_{n-1} \frac{d^{n-1} q}{dt^{n-1}} + \ldots + a_1 \frac{dq}{dt} + a_o q$$

$$= C_m \frac{d^m F}{dt^m} + C_{m-1} \frac{d^{m-1} F}{dt^{m-1}} + \ldots + C_1 \frac{dF}{dt} + C_o F$$

can be calculated from the graphical knowledge of $q(t)$ and $F(t)$. It is noticed that the method may become somewhat cumbersome if the input $F(t)$ is not of certain types. The input which allows the method to be applied most easily is one which goes to zero quickly, giving free oscillation data. A simplification also occurs when the input $F(t)$ is known accurately to be a function of a certain type, such as a step, a ramp $F(t) = t$ or, perhaps, something of the form $\sum_i A_i e^{\mu_i t}$

The method has as its primary advantages first, the fact that the correct quantity (the sum of the squares of the differences between the calculated and the measured quantities) is minimized. The ordinary simple equations of least squares may be used in this minimization, since the equations of condition (the redundant or inconsistent equations to be solved by least squares) satisfy the restrictions under which the least squares solution is normally derived. That is, the equations are linear, and only the right-hand sides are subject to error. The so-called "derivative method" used in example III minimizes

$\Sigma(\ddot{q} + b\dot{q} + kq - C_1\dot{F} - C_0F)^2$, a quantity with little, if any meaning. Other methods, Prony's, the Laplace and Fourier methods, are subject to the same objection.

Another advantage of the method is that the data are analyzed directly in the time plane, thus eliminating the possibility of the introduction of errors due to graphical integration (in the case of Fourier and Laplace transform methods) or differentiation (derivative method). Where the inputs give free oscillation data there is a certain saving of time due to this directness, no preliminary steps having to be taken before the method can be applied.

For the usual type of problem which is found, those with the pulse- or step-type inputs, the method is at least as rapid as any other which gives comparable results.

Ames Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Moffett Field, Calif., Jan. 17, 1951.

# APPENDIX A

## PRONY'S METHOD

A problem which occurs quite often is that of fitting a function $q(t)$ to a sum of exponentials. One method used to fit a function of this form is known as Prony's method and is described below. (See also references 1 and 3.)

It is first assumed that the data are given in tabular form with equal intervals of the argument $t$. Suppose measurements of $q(t)$ are taken at instants $t_0, t_1, \ldots, t_{\nu-1}$, $\nu > 2n$, where $n$ is the number of exponentials to which $q$ is to be fitted. Then it is assumed that $t_k = t_0 + k(\Delta t)$, $\Delta t = \text{constant}$, $k = 0, 1, \ldots, \nu-1$. By a proper shift of the time axis, $t_0$ may be taken equal to zero. Let $q_k = q(t_k) = q(k\Delta t)$. Fundamental to Prony's method is the following theorem: If

$$q_k = \sum_{i=1}^{n} B_i \, e^{\lambda_i t_k} \tag{19}$$

$k = 0, 1, \ldots, \nu-1$, then $q$ satisfies the linear difference equation

$$q_{k+n} + Q_{n-1} \, q_{k+n-1} + Q_{n-2} \, q_{k+n-2} + \ldots + Q_0 q_k = 0 \tag{20}$$

where $Q_0, \ldots, Q_{n-1}$ are constants such that the roots of the equation

$$x^n + Q_{n-1} \, x^{n-1} + \ldots Q_0 = 0 \tag{21}$$

are $e^{\lambda_i(\Delta t)}$, $i = 1, \ldots, n$. The proof is as follows:

$$q_{k+n} + Q_{n-1} \, q_{k+n-1} + \ldots + Q_0 q_k = \sum_{i=1}^{n} B_i \, e^{\lambda_i (k+n)\Delta t} +$$

$$Q_{n-1} \sum_{i=1}^{n} B_i \, e^{\lambda_i (k+n-1)\Delta t} + \ldots + Q_0 \sum_{i=1}^{n} B_i \, e^{\lambda_i k\Delta t}$$

$$= \sum_{i=1}^{n} B_i \, e^{\lambda_i \Delta t} \left[ \left(e^{\lambda_i \Delta t}\right)^n + Q_{n-1} \left(e^{\lambda_i \Delta t}\right)^{n-1} + \ldots + Q_0 \right]$$

Certainly there exist $n$ constants $Q_o, \ldots, Q_{n-1}$, such that $e^{\lambda_1 \Delta t}, e^{\lambda_2 \Delta t}, \ldots, e^{\lambda_n \Delta t}$ are the roots of equation (21). Choosing these constants in this way then makes the bracketed expression above and, therefore, $q_{k+n} + Q_{n-1} q_{k+n-1} + \ldots + Q_o$ vanish. This completes the proof.

Prony's method consists of writing down the equations

$$q_n + Q_{n-1} q_{n-1} + \ldots + Q_o q_o = 0$$

$$q_{n+1} + Q_{n-1} q_n + \ldots + Q_o q_1 = 0$$

$$q_{n+2} + Q_{n-1} q_{n+1} + \ldots + Q_o q_2 = 0$$

$$\cdots\cdots$$

$$\cdots\cdots$$

and solving them by least squares[2] for $Q_i$, $i = 0, \ldots, n-1$. From the normal equations obtained from the least-squares process, $Q_o, \ldots, Q_{n-1}$ can be found. Then the roots of the equation

$$x^n + Q_{n-1} x^{n-1} + \ldots + Q_o = 0$$

are calculated, giving $e^{\lambda_i (\Delta t)}$ and, therefore, $\lambda_i$.

[2]Herein lies one of the objections to Prony's method, since

$$\sum_{k=0}^{\nu-n} \left(q_{k+n} + Q_{n-1} q_{k+n-1} + \ldots + Q_o q_k\right)^2$$

rather than

$$\sum_{k=0}^{\nu-1} \left(q_k - \sum_{i=1}^{n} B_i e^{\lambda_i t_k}\right)_1$$

is minimized, the correct minimization procedure leading, in this case, to a forbidding amount of calculation.

The coefficient $e^{\lambda_i t}$ of $B_i$ in equation (19) can be tabulated for each $i$, since $\lambda_i$ is known. Therefore, $B_i$ can be found by a second least-squares procedure. This completes Prony's method.

The objection described in the preceding footnote can be overcome by considering the $\lambda_i$ and $B_i$ found by Prony's method not as the best possible values of these parameters, but only as a first approximation thereof. A better approximation can then be found by means of a Taylor's series. This method is described in the body of this report.

## Extension to Prony's Method

Suppose now that it is required to find some way to fit a function to a sum of exponentials plus an (unknown) constant. The solution will be presented herein for the case in which the exponentials are two in number, but the generalization to a greater number of unknowns will be evident.

To be specific, a quantity $q$ must be fitted to a function of the form

$$q = B_1 e^{\lambda_1 t} + B_2 e^{\lambda_2 t} + B_3 \tag{22}$$

$q$ is first fitted to a function of the form

$$q = B_1 e^{\lambda_1 t} + B_2 e^{\lambda_2 t} + B_3 e^{\lambda_3 t} \tag{23}$$

and the condition that $\lambda_3 = 0$ is put in later. Prony's method is applied first to equation (2). As before $q$ satisfies the difference equation

$$q_{k+3} + Q_2 q_{k+2} + Q_1 q_{k+1} + Q_0 q_k = 0 \tag{24}$$

where the constants $Q_i$, i=0, 1, 2, 3, are such that the equation

$$x^3 + Q_2 x^2 + Q_1 x + Q_0 = 0 \tag{25}$$

has $e^{\lambda_i(\Delta t)}$ as roots. But $\lambda_3 = 0$. Therefore, equation (4) has unity as a root, and $1 + Q_2 + Q_1 + Q_0 = 0$. Eliminating $Q_0$ (say) between this and equation (24)

$$(q_{k+2} - q_k)\ Q_2 + (q_{k+1} - q_k)\ Q_1 = (q_k - q_{k+3}) \tag{26}$$

is obtained. ~~This~~ equation is now solved by least squares for $Q_1$ and $Q_2$. The exponentials $e^{\lambda_1(\Delta t)}$ and $e^{\lambda_2(\Delta t)}$ are now found to be the roots of $x^2 + (Q_2 + 1)\ x + (Q_1 + Q_2 + 1) = 0$. Prony's method now proceeds as before.

An evident extension of this method may be used to simplify the problem of fitting a sum of exponentials when one or more of the exponents are already known.

## Example of Prony's Method

Suppose the data found in column (3), table VII is to be fitted to two exponentials (see example I of the body of this report). In this case, equations (20) become

$$(5) + (4)\ Q_1 + (3)\ Q_0 = 0$$

Solving by least squares,

$$\begin{cases} Q_1\ \Sigma\ (4)^2 + Q_0\ \Sigma\ (3)\ (4) = -\ \Sigma\ (4)\ (5) \\ Q_1\ \Sigma\ (3)\ (4) + Q_0\ \Sigma\ (3)^2 = -\ \Sigma\ (3)\ (5) \end{cases}$$

which give

$$Q_0 = 0.8320$$

$$Q_1 = -\ 1.3922$$

Equation (21) thus becomes

$$x^2 - 1.3922\ x + 0.8320 = 0$$

or

$$x = e^{\lambda_1(\Delta t)} = 0.6961 + 0.5895\ i$$

Therefore,

$$\lambda_1 = \frac{1}{\Delta t}\ ln\ (0.6961 + 0.5895\ i)$$

$$= 10\left[ln\ \sqrt{(0.6961)^2+(0.5895)^2} + i\ \text{arc tan}\ \frac{0.5895}{0.6961}\right]$$

$$= -\ 0.91 + 7.02\ i$$

$$\lambda_2 = -\ 0.91 - 7.02\ i$$

Since $\lambda_1$ and $\lambda_2$ are complex conjugate, so are $B_1$ and $B_2$. Let $B_1 = \beta + \beta' i$. Then $q = 2\ e^{lt}\ (\beta \cos l't - \beta' \sin l't)$. Referring to table VII, this becomes (7) $\beta$ − (8) $\beta'$ = (12) . Solving as before by least squares,

$$\beta = 0.3681$$

$$\beta' = -\ 2.6775$$

# APPENDIX B

## DISCONTINUOUS INPUTS

Consider the differential equation

$$\frac{d^n q}{dt^n} + a_{n-1}\frac{d^{n-1}q}{dt^{n-1}} + \dots + a_1\frac{dq}{dt} + a_0 q = C_m\frac{d^m F}{dt^m} + \dots + C_1\frac{dF}{dt} + C_0 F \tag{2}$$

A fundamental question which often arises is the following. Suppose $F(t)$ or one of its derivatives is discontinuous at a point. At this point all higher derivatives fail to exist. What is the meaning, if any, of the differential equation (2) at this point?

An instance of this problem occurs in example II of the present report. There, $F(t)$ is a step: $F(t) = \begin{cases} 0, & t<0 \\ 1, & t>0 \end{cases}$. At $t = 0$ $dF/dt$ and all higher derivatives of $F$ do not exist.

Throughout the following discussion, let $x(0+) = \lim\limits_{t \to 0+} x(t)$, where $x(t)$ is any function, and $t \to 0+$ indicate that $t$ is to approach zero through positive values only. Consider again the differential equation

$$\ddot{q} + a_1\dot{q} + a_0 q = C_1\dot{F} + C_0 F \tag{16}$$

where dots denote differentiation with respect to $t$. Integrate equation (16) between the limits $-\epsilon$ and $t$ $(\epsilon > 0)$ to obtain

$$[\dot{q}(t) - \dot{q}(-\epsilon)] + a_1[q(t) - q(-\epsilon)] + a_0\int_{-\epsilon}^{t} q(t)dt$$

$$= C_1\,[F(t) - F(-\epsilon)] + C_0\int_{-\epsilon}^{t} F(t)dt$$

It is now assumed that $\int_{-\epsilon}^{t} q(t)dt$ is continuous. (In all examples used in this report, $q(t)$ represents the pitching velocity of the airplane, making $\int_{-\epsilon}^{t} q(t)dt$ the angle of pitch which, from physical considerations, may be seen to be continuous.)

$$\int_{-\epsilon}^{t} F(t)dt = \int_{0}^{t} F(t)dt = t, \text{ since } F(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

Letting $\epsilon \longrightarrow 0$,

$$[\dot{q}(t)-\dot{q}(0-)] + a_1[q(t)-q(0-)] + k[\theta(t)-\theta(0-)] = C_1F(t) + C_0t$$

where

$$\theta(t) = \int_0^t q(t)\,dt$$

Integrating once more with respect to $t$, this time from $-\epsilon$ to $+\epsilon$,

$$[q(\epsilon)-q(-\epsilon)-2\epsilon\dot{q}(0-)] + a_1 \; [\theta(\epsilon)-\theta(-\epsilon)-2\epsilon q(0-)] \; +$$

$$a_0\left[\int_{-\epsilon}^{t} \theta(t)dt - 2\epsilon\theta(0-)\right] = C_1\,\epsilon$$

Again letting $\epsilon \longrightarrow 0$, it is seen that $q(0+) - q(0-) = 0$, since $\theta(t)$ was assumed continuous. Thus $q(t)$ is continuous at zero. Going back to equation (19) and letting $t \longrightarrow 0$, $\dot{q}(0+) - \dot{q}(0-) = C_1F(0+) = C_1$. Thus, $\dot{q}(t)$ is discontinuous at zero, and the difference between the right-hand and left-hand limits of $\dot{q}(t)$ as $t$ approaches zero is $C_1$. Thus, writing equation (4a) in terms of $\dot{q}(0-)$ rather than $q(0+)$,

$$q(t) = \frac{\lambda_1 q(0-)-\lambda_1\lambda_2 q(0)+C_1\lambda_1+C_0}{\lambda_1(\lambda_1-\lambda_2)}\,e^{\lambda_1 t} + \frac{\lambda_2 q(0-)-\lambda_1\lambda_2 q(0)+C_1\lambda_2+C_0}{\lambda_2(\lambda_2-\lambda_1)}\,e^{\lambda_2 t} +$$

$$\frac{C_0}{\lambda_1\lambda_2}$$

and $q(t)$ may be seen after all to be dependent on $C_1$.

## REFERENCES


1. Greenberg, Harry: A Survey of Methods for Determining Stability Parameters of an Airplane from Dynamic Flight Measurements. NACA TN 2340, 1951.

2. Ince, E. L.: Ordinary Differential Equations. Dover, New York, N. Y., 1944.

3. Whittaker, Sir Edmund, and Robinson, G.: The Calculus of Observations. Blackie and Son Ltd., London, Eng., 1944.

4. Booth, A. D.: An Application of the Method of Steepest Descents to the Solution of Systems of Nonlinear Simultaneous Equations. The Quart. Jour. of Mech. and App. Math., Dec. 1949.

5. Conkwright, M. B.: Introduction to the Theory of Equations. Ginn and Co., New York, N. Y., 1941.

TABLE I.— THE TAYLOR'S SERIES ITERATION APPLIED TO EXAMPLE I

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Row | t | $q_m(t)$ | 1/2 (3) | 0.91t | $e^{(5)}$ | (4)×(6) | 7.02t | cos (8) | sin (8) | 0.3681(9) | 0.3681(10) | 2.6775(9) | 2.6775 (10) | (11)+(14) | (12)−(13) | (2)× (15) | (2) × (16) | (7)−(15) |
| 1 | 0.4 | 0.7539 | 0.37695 | 0.364 | 1.43907 | 0.54246 | 2.808 | -0.94495 | 0.32722 | -0.34784 | 0.12045 | -2.53010 | 0.87613 | 0.52829 | 2.65055 | 0.21132 | 1.06022 | 0.01417 |
| 2 | .5 | -1.6600 | -.83000 | .455 | 1.57617 | -1.30822 | 3.510 | - .93283 | -.36032 | -.34337 | -.13263 | -2.49765 | -.96476 | -1.30813 | 2.36502 | -.65407 | 1.18251 | -.00009 |
| 3 | .6 | -2.9386 | -1.46930 | .546 | 1.72633 | -2.53650 | 4.212 | -.47946 | -.87756 | -.17649 | -.32303 | -1.28375 | -2.34967 | -2.52616 | .96072 | -1.51570 | .57643 | -.01034 |
| 4 | .7 | -2.7099 | -1.35495 | .637 | 1.89080 | -2.56194 | 4.914 | .20057 | -.97968 | .07383 | -.36062 | .53703 | -2.62309 | -2.54926 | -.89765 | -1.78448 | -.62836 | -.01268 |
| 5 | .8 | -1.3279 | -.66395 | .728 | 2.07093 | -1.37499 | 5.616 | .78586 | -.61841 | .28928 | -.22764 | 2.10414 | -1.65579 | -1.36651 | -2.33178 | -1.09321 | -1.86542 | -.00848 |
| 6 | .9 | .4055 | .20275 | .819 | 2.26823 | .45988 | 6.318 | .99938 | .03525 | .36787 | .01298 | 2.67584 | .09438 | .46225 | -2.66286 | .41603 | -2.39657 | -.00237 |
| 7 | 1.0 | 1.6693 | .83465 | .910 | 2.48432 | 2.07364 | 7.020 | .74022 | .67237 | .27247 | .24750 | 1.98194 | 1.80027 | 2.07274 | -1.73444 | 2.07274 | -1.73444 | .00080 |
| 8 | 1.1 | 1.9868 | .99340 | 1.001 | 2.72100 | 2.70304 | 7.722 | .13105 | .99138 | .04824 | .36493 | .35089 | 2.65442 | 2.70266 | .01404 | 2.97293 | .01544 | .00038 |
| 9 | 1.2 | 1.3773 | .68865 | 1.092 | 2.98023 | 2.05234 | 8.424 | -.54024 | .84151 | -.19886 | .30976 | -1.44649 | 2.25314 | 2.05428 | 1.75625 | 2.46514 | 2.10750 | -.00194 |
| 10 | 1.3 | .2644 | .13220 | 1.183 | 3.26415 | .43152 | 9.126 | -.95590 | .29371 | -.35187 | .10811 | -2.55942 | .78641 | .43454 | 2.66753 | .56490 | 3.46779 | -.00302 |
| 11 | 1.4 | -.7775 | -.38875 | 1.274 | 3.57512 | -1.38983 | 9.828 | -.91955 | -.39298 | -.33849 | -.14466 | -2.46210 | -1.05220 | -1.39069 | 2.31744 | -1.94697 | 3.24442 | .00086 |
| 12 | 1.5 | -1.3026 | -.65130 | 1.365 | 3.91572 | -2.55031 | 10.530 | -.44823 | -.89392 | -.16499 | -.32905 | -1.20014 | -2.39347 | -2.55846 | .87109 | -3.83769 | 1.30664 | .00815 |
| 13 | 1.6 | -1.1666 | -.58330 | 1.456 | 4.28877 | -2.50164 | 11.232 | .23497 | -.97200 | .08649 | -.35779 | .62913 | -2.60253 | -2.51604 | -.98692 | -4.02566 | -1.57907 | .01440 |
| 14 | 1.7 | -.5405 | -.27025 | 1.547 | 4.69736 | -1.26946 | 11.934 | .80717 | -.59032 | .29712 | -.21730 | 2.16120 | -1.58058 | -1.28346 | -2.37850 | -2.18188 | -4.04345 | .01400 |
| 15 | 1.8 | .2181 | .10905 | 1.638 | 5.14487 | .56105 | 12.636 | .99752 | .07045 | .36719 | .02593 | 2.67086 | .18863 | .55582 | -2.64493 | 1.00048 | -4.76087 | .00523 |
| 16 | 1.9 | .7532 | .37660 | 1.729 | 5.63502 | 2.12215 | 13.338 | .71606 | .69804 | .26358 | .25695 | 1.91725 | 1.86900 | 2.13258 | -1.66030 | 4.05190 | -3.15457 | -.01043 |
| 17 | 2.0 | .8672 | .43360 | 1.820 | 6.17186 | 2.67612 | 14.040 | .09602 | .99538 | .03534 | .36640 | .25709 | 2.66513 | 2.70047 | .10931 | 5.40094 | .21862 | -.02435 |

$\Sigma\ (17)^2 = 112.9270614658$

$\Sigma (17)(18) = -8.1710008924$

$\Sigma (9)(17) = 5.6094938638$

$\Sigma (10)(17) = 28.3267018086$

$\Sigma (17)(19) = -0.2440723067$

$\Sigma\ (18)^2 = 95.6899969052$

$\Sigma (9)(18) = -26.2944439157$

$\Sigma (10)(18) = 1.8997789841$

$\Sigma (18)(19) = -0.0407482910$

$\Sigma\ (9)^2 = 8.6904772300$

$\Sigma (9)(10) = 0.4155203282$

$\Sigma (9)(19) = -0.0096968655$

$\Sigma\ (10)^2 = 8.3095740806$

$\Sigma (10)(19) = -0.0313131874$

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TABLE II.- FURTHER CALCULATIONS NECESSARY FOR EXAMPLE I

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| Row | t | 0.92t | $e^{-.92t}$ | 7.0252t | cos (5) | sin (5) | F(t) | (4)(6)(8) | (4)(7)(8) |
| 1 | 0 | 0 | 1.0000 | 0 | 1.0000 | 0 | 0 | 0 | 0 |
| 2 | .05 | .046 | .9550 | .3513 | .9389 | .3441 | .05 | .04483 | .0164 |
| 3 | .1 | .092 | .9121 | .7025 | .7632 | .6461 | .1 | .06961 | .0589 |
| 4 | .15 | .138 | .8711 | 1.0538 | .4943 | .8693 | .15 | .06469 | .1136 |
| 5 | .2 | .184 | .8319 | 1.4050 | .1650 | .9863 | .2 | .02745 | .1641 |
| 6 | .25 | .230 | .7945 | 1.7563 | -.1844 | .9828 | .15 | -.02198 | .1171 |
| 7 | .3 | .276 | .7588 | 2.1075 | -.5113 | .8594 | .1 | -.03880 | .0652 |
| 8 | .35 | .322 | .7247 | 2.4588 | -.7758 | .6310 | .05 | -.02811 | .0229 |
| 9 | .4 | .368 | .6921 | 2.8100 | -.9455 | .3256 | 0 | 0 | 0 |

TABLE III.– TAYLOR'S SERIES ITERATION APPLIED TO EXAMPLE II

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Row | t | $q_m(t)/2$ | 0.91t | 7t | $e^{-(4)}$ | cos (5) | sin (5) | (6)×(7) | (6)×(8) | 1.11 (9) | 9.7 (10) | 1.11 (10) | 9.7 (9) | (9) −1 | (12) − (11) | (13) + (14) | (2) × (16) | (2) × (17) | (3) − (16) | (20) − 1.11 |
| 1 | 0 | 0 | 0 | 0 | 1.00000 | 1.00000 | 0 | 1.00000 | 0 | 1.1100 | 0 | 0 | 9.70 | 0 | −1.11 | 9.70 | 0 | 0 | 1.11 | 0 |
| 2 | .1 | 5.87880 | .0910 | .7 | .91302 | .76481 | .64426 | .69829 | .58822 | .77510 | 5.70573 | .65292 | 6.77341 | −.30171 | 4.93063 | 7.42633 | .49306 | .74263 | .94817 | −.16183 |
| 3 | .2 | 8.68640 | .1820 | 1.4 | .83360 | .16987 | .98547 | .14160 | .82149 | .15718 | 7.96845 | .91185 | 1.37352 | −.85840 | 7.81127 | 2.28537 | 1.56225 | .45707 | .87513 | −.23487 |
| 4 | .3 | 7.70305 | .2730 | 2.1 | .76109 | −.50498 | .86313 | −.38434 | .68692 | −.42662 | 6.37212 | .72918 | −3.72810 | −1.38434 | 6.79874 | −2.99892 | 2.03962 | −.89968 | .90431 | −.20569 |
| 5 | .4 | 3.99835 | .3640 | 2.8 | .69489 | −.94229 | .33479 | −.65479 | .23264 | −.72682 | 2.25661 | .25823 | −6.35146 | −1.65479 | 2.98343 | −6.09323 | 1.19337 | −2.43729 | 1.01492 | −.09608 |
| 6 | .5 | −.34015 | .4550 | 3.5 | .63445 | −.93637 | −.35102 | −.59408 | −.22270 | −.65943 | −2.16019 | −.24720 | −5.76258 | −1.59408 | −1.50076 | −6.00978 | −.75038 | −3.00489 | 1.16061 | .05061 |
| 7 | .6 | −3.29970 | .5460 | 4.2 | .57926 | −.48999 | −.87173 | −.28383 | −.50496 | −.31505 | −4.89811 | −.56051 | −2.75315 | −1.28383 | −4.58306 | −3.31366 | −2.74984 | −1.98820 | 1.28336 | .17336 |
| 8 | .7 | −3.80935 | .6370 | 4.9 | .52888 | .18687 | −.98239 | .098832 | −.51957 | .10970 | −5.03983 | −.57672 | .95867 | −.90117 | −5.14953 | .38195 | −3.60467 | .26737 | 1.34018 | .23018 |
| 9 | .8 | −2.05715 | .7280 | 5.6 | .48287 | .77583 | −.63095 | .37463 | −.30467 | .41584 | −2.95530 | −.33818 | 3.63391 | −.62537 | −3.37114 | 3.29573 | −2.69691 | 2.63662 | 1.31399 | .20399 |
| 10 | .9 | .80590 | .8190 | 6.3 | .44087 | .99985 | .01728 | .44080 | .0076182 | .48929 | .073897 | .0084562 | 4.27576 | −.55920 | −.41539 | 4.28422 | −.37385 | 3.85580 | 1.22129 | .11129 |
| 11 | 1.0 | 3.33430 | .9100 | 7.0 | .40252 | .75356 | .65738 | .30332 | .26461 | .33669 | 2.56672 | .29372 | 2.94220 | −.69668 | 2.23003 | 3.23592 | 2.23003 | 3.23592 | 1.10427 | −.00573 |
| 12 | 1.1 | 4.47300 | 1.0010 | 7.7 | .36751 | .15281 | .98826 | .056159 | .36320 | .062336 | 3.52304 | .40315 | .54474 | −.94384 | 3.46070 | .94789 | 3.80677 | 1.04268 | 1.01230 | −.09770 |
| 13 | 1.2 | 3.95430 | 1.0920 | 8.4 | .33554 | −.51982 | .85428 | −.17442 | .28665 | −.19361 | 2.78051 | .31818 | −1.69187 | −1.17442 | 2.97412 | −1.37369 | 3.56894 | −1.64843 | .98018 | −.12982 |
| 14 | 1.3 | 2.28475 | 1.1830 | 9.1 | .30636 | −.94794 | .31846 | −.29041 | .097563 | −.32236 | .94636 | .10829 | −2.81698 | −1.29041 | 1.26872 | −2.70869 | 1.64934 | −3.52130 | 1.01603 | −.09397 |
| 15 | 1.4 | .39245 | 1.2740 | 9.8 | .27971 | −.93016 | −.36715 | −.26018 | −.10270 | −.28880 | −.99619 | −.11400 | −2.52375 | −1.26018 | −.70739 | −2.63775 | −.99035 | −3.69285 | 1.09984 | −.01016 |
| 16 | 1.5 | −.85365 | 1.3650 | 10.5 | .25538 | −.47486 | −.88006 | −.12127 | −.22475 | −.13461 | −2.18008 | −.24947 | −1.17632 | −1.12127 | −2.04547 | −1.42579 | −3.06821 | −2.13869 | 1.19182 | .08182 |
| 17 | 1.6 | −1.01375 | 1.4560 | 11.2 | .23317 | .20381 | −.97901 | .047522 | −.22828 | .052749 | −2.21432 | −.25339 | .46096 | −.95248 | −2.26707 | .20757 | −3.62731 | .33211 | 1.25332 | .14332 |
| 18 | 1.7 | −.20035 | 1.5470 | 11.9 | .21289 | .78661 | −.61745 | .16746 | −.13145 | .18588 | −1.27507 | −.14591 | 1.62436 | −.83254 | −1.46095 | 1.47845 | −2.48362 | 2.51337 | 1.26060 | .15060 |
| 19 | 1.8 | 1.06570 | 1.6380 | 12.6 | .19437 | .99940 | .03455 | .19425 | .0067155 | .21562 | .065140 | .0074542 | 1.88423 | −.80575 | −.15048 | 1.89168 | −.27086 | 3.40502 | 1.21618 | .10618 |
| 20 | 1.9 | 2.15110 | 1.7290 | 13.3 | .17746 | .74209 | .67030 | .13169 | .11895 | .14618 | 1.15382 | .13203 | 1.27739 | −.86831 | 1.00764 | 1.40942 | 1.91462 | 2.67790 | 1.14346 | .03346 |
| 21 | 2.0 | 2.60945 | 1.8200 | 14.0 | .16203 | .13572 | .99075 | .021991 | .16053 | .024410 | 1.55714 | .17819 | .21331 | −.97801 | 1.53273 | .39150 | 3.06546 | .78300 | 1.67672 | −.03328 |

$\Sigma\,(18)^2 = 114.5777739186$

$\Sigma\,(18)\,(19) = 3.4142472252$

$\Sigma\,(15)\,(18) = -2.1151600764$

$\Sigma\,(10)\,(18) = 13.25873312032$

$\Sigma\,(18)\,(21) = -5.1204775702$

$\Sigma\,(19)^2 = 113.0750629234$

$\Sigma\,(15)\,(19) = 10.0084494394$

$\Sigma\,(10)\,(19) = 1.75430204377$

$\Sigma\,(19)\,(21) = 1.8592250271$

$\Sigma\,(15)^2 = 22.4137740234$

$\Sigma\,(10)\,(15) = -1.066329105095$

$\Sigma\,(15)\,(21) = -0.0609152759$

$\Sigma\,(10)^2 = 2.63796281518049$

$\Sigma\,(10)\,(21) = -0.8789726095$

TABLE IV.— SECOND ITERATION FOR EXAMPLE II

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Row | t | $q_m(t)/2$ | 0.92t | 7.0242t | $e^{-④}$ | cos ⑤ | sin ⑤ | ⑥ × ⑦ | ⑥ × ⑧ | 1.1392 ⑨ | 9.3890 ⑩ | 1.1392 ⑩ | 9.3890 ⑨ | ⑨ − 1 | ⑫ − ⑪ | ⑬ + ⑭ | ② × ⑯ | ② × ⑰ | ③ − ⑯ | ⑳ −1.1392 |
| 1 | 0 | 0 | 0 | 0 | 1.0 | 1.0 | 0 | 1.0 | 0 | 1.1392 | 0 | 0 | 9.3890 | 0 | −1.1392 | 9.3890 | 0 | 0 | 1.13920 | 0 |
| 2 | .1 | 5.87880 | .092 | .70242 | .91211 | .76323 | .64612 | .69615 | .58933 | .79305 | 5.53322 | .67136 | 6.53615 | −0.30385 | 4.74017 | 7.20751 | .47402 | .72075 | 1.13863 | −.00057 |
| 3 | .2 | 8.68640 | .184 | 1.40484 | .83194 | .16505 | .98629 | .13731 | .82053 | .15642 | 7.70396 | .93475 | 1.28920 | −.86269 | 7.54754 | 2.22395 | 1.50951 | .44479 | 1.13886 | −.00034 |
| 4 | .3 | 7.70305 | .276 | 2.10726 | .75881 | −.51129 | .85941 | −.38797 | .65213 | −.44198 | 6.12285 | .74291 | −3.64265 | −1.38297 | 6.56483 | −2.89974 | 1.96945 | −.86992 | 1.13822 | −.00098 |
| 5 | .4 | 3.99835 | .368 | 2.80968 | .69212 | −.94546 | .32573 | −.65437 | .22544 | −.74546 | 2.11666 | .25682 | −6.14388 | −1.65437 | 2.86212 | −5.88706 | 1.14485 | −2.35482 | 1.13623 | −.00297 |
| 6 | .5 | −.34015 | .460 | 3.51210 | .63128 | −.93207 | −.36228 | −.58840 | −.22870 | −.67031 | −2.14726 | −.26054 | −5.52449 | −1.58840 | −1.47695 | −5.78503 | −.73848 | −2.89252 | 1.13680 | −.00240 |
| 7 | .6 | −3.29970 | .552 | 4.21452 | .57580 | −.47731 | −.87873 | −.27484 | −.50597 | −.31310 | −4.75055 | −.57640 | −2.58047 | −1.27484 | −4.43745 | −3.15687 | −2.66247 | −1.89412 | 1.13775 | −.00145 |
| 8 | .7 | −3.80935 | .644 | 4.91694 | .52519 | .20347 | −.97908 | .10686 | −.51420 | .12173 | −4.82782 | −.58578 | 1.00331 | −.89314 | −4.94955 | .41753 | −3.46469 | .29227 | 1.14020 | .00100 |
| 9 | .8 | −2.05715 | .736 | 5.61936 | .47903 | .78790 | −.61580 | .37743 | −.29499 | .42997 | −2.76966 | −.33605 | 3.54369 | −.62287 | −3.19963 | 3.20764 | −2.55970 | 2.56611 | 1.14248 | .00328 |
| 10 | .9 | .80590 | .828 | 6.32178 | .43692 | .99924 | .03909 | .43659 | .015554 | .49736 | .14604 | .017719 | 4.09914 | −.56341 | −.35132 | 4.11686 | −.31619 | 3.70517 | 1.15722 | .01802 |
| 11 | 1.0 | 3.33430 | .920 | 7.02420 | .39852 | .73740 | .67546 | .29387 | .26918 | .33478 | 2.52733 | .30665 | 2.75915 | −.70613 | 2.19255 | 3.06580 | 2.19255 | 3.06580 | 1.14175 | .00255 |
| 12 | 1.1 | 4.47300 | 1.012 | 7.72662 | .36349 | .12637 | .99198 | .045934 | .36057 | .052328 | 3.38539 | .41076 | .43127 | −.95407 | 3.33306 | .84203 | 3.66637 | .92623 | 1.13994 | .00074 |
| 13 | 1.2 | 3.95430 | 1.104 | 8.42904 | .33154 | −.54435 | .83886 | −.18047 | .27812 | −.20559 | 2.61127 | .31683 | −1.69443 | −1.18047 | 2.81686 | −1.37760 | 3.38023 | −1.65312 | 1.13744 | −.00176 |
| 14 | 1.3 | 2.28475 | 1.196 | 9.13146 | .30240 | −.95747 | .28853 | −.28954 | .087251 | −.32984 | .81920 | .099396 | −2.71849 | −1.28954 | 1.14904 | −2.61909 | 1.49375 | −3.40482 | 1.13571 | −.00349 |
| 15 | 1.4 | .39245 | 1.288 | 9.83388 | .27582 | −.91720 | −.39843 | −.25298 | −.10989 | −.28819 | −1.03176 | −.12519 | −2.37523 | −1.25298 | −.74357 | −2.50042 | −1.04100 | −3.50059 | 1.13602 | −.00318 |
| 16 | 1.5 | −.85365 | 1.380 | 10.53630 | .25158 | −.44260 | −.89672 | −.11135 | −.22560 | −.12685 | −2.11816 | −.25700 | −1.04547 | −1.11135 | −1.99131 | −1.30247 | −2.98697 | −1.95371 | 1.13766 | −.00154 |
| 17 | 1.6 | −1.01375 | 1.472 | 11.23872 | .22947 | .24158 | −.97038 | .055435 | −.22287 | .063152 | −2.09253 | −.25389 | .52048 | −.94457 | −2.15568 | .26659 | −3.44909 | .42654 | 1.14193 | .00273 |
| 18 | 1.7 | −.20035 | 1.564 | 11.94114 | .20930 | .81137 | −.58453 | .16982 | −.12234 | .19346 | −1.14865 | −.13937 | 1.59444 | −.83018 | −1.34211 | 1.45507 | −2.28159 | 2.47362 | 1.14176 | .00256 |
| 19 | 1.8 | 1.06570 | 1.656 | 12.64356 | .19090 | .99694 | .07811 | .19032 | .014911 | .21681 | .14000 | .016987 | 1.78691 | −.80968 | −.07681 | 1.80390 | −.13826 | 3.24702 | 1.14251 | .00331 |
| 20 | 1.9 | 2.15110 | 1.748 | 13.34598 | .17412 | .71055 | .70364 | .12372 | .12252 | .14094 | 1.15034 | .13957 | 1.16161 | −.87628 | 1.00940 | 1.30118 | 1.91786 | 2.47224 | 1.14170 | .00250 |
| 21 | 2.0 | 2.60945 | 1.840 | 14.04840 | .15882 | .08768 | .99615 | .013925 | .15821 | .015863 | 1.48543 | .18023 | .13074 | −.98608 | 1.46957 | .31097 | 2.93914 | .62194 | 1.13988 | .00068 |

$\Sigma$ ⑱$^2$ = 105.3331271177

$\Sigma$ ⑱ ⑲ = 2.9595744569

$\Sigma$ ⑮ ⑱ = −2.2940120183

$\Sigma$ ⑩ ⑱ = 12.6247313343

$\Sigma$ ⑱ ㉑ = −0.0219082303

$\Sigma$ ⑲$^2$ = 103.3686471536

$\Sigma$ ⑮ ⑲ = 9.5493208522

$\Sigma$ ⑩ ⑲ = 1.7416535628

$\Sigma$ ⑲ ㉑ = 0.1547332570

$\Sigma$ ⑮$^2$ = 22.3984776563

$\Sigma$ ⑩ ⑮ = −1.072452563

$\Sigma$ ⑮ ㉑ = −0.0011647190

$\Sigma$ ⑩$^2$ = 2.611953303938

$\Sigma$ ⑩ ㉑ = −0.00144476960

TABLE V.– TABULATION OF $F(t)$, $\dot{F}(t)$, $q(t)$, $\dot{q}(t)$, AND $\ddot{q}(t)$ FOR EXAMPLE III

| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| | $t$ | $F$ | $\dot{F}$ | $q$ | $\dot{q}$ | $\ddot{q}$ |
| 1 | 0 | 0 | 7.03 | 0 | 0 | 941 |
| 2 | .1 | .5893 | 4.35 | 3.940 | 68.4 | 327 |
| 3 | .2 | .8206 | .21 | 10.939 | 57.1 | –532 |
| 4 | .3 | .6521 | –3.33 | 12.974 | –23.9 | –978 |
| 5 | .4 | .2251 | –4.81 | 5.847 | –113.9 | –702 |
| 6 | .5 | –.2290 | –3.92 | –7.837 | –146.1 | 110 |
| 7 | .6 | –.5062 | –1.46 | –20.407 | –91.3 | 938 |
| 8 | .7 | –.5141 | 1.23 | –24.019 | 24.6 | 1266 |
| 9 | .8 | –.2947 | 2.92 | –15.576 | 137.9 | 886 |
| 10 | .9 | .0175 | 3.05 | 1.300 | 184.8 | 5 |
| 11 | 1.0 | .2694 | 1.81 | 18.214 | 138.3 | –895 |
| 12 | 1.1 | .3606 | –.01 | 26.577 | 21.5 | –1334 |
| 13 | 1.2 | .2779 | –1.53 | 22.181 | –105.4 | –1092 |
| 14 | 1.3 | .0869 | –2.11 | 7.319 | –178.7 | –312 |
| 15 | 1.4 | –.1103 | –1.67 | –10.555 | –163.6 | 594 |
| 16 | 1.5 | –.2258 | –.57 | –22.775 | –71.0 | 1171 |
| 17 | 1.6 | –.2226 | .60 | –23.785 | 50.6 | 1155 |
| 18 | 1.7 | –.1221 | 1.31 | –13.711 | 141.4 | 589 |
| 19 | 1.8 | .0152 | 1.32 | 2.053 | 160.2 | –219 |
| 20 | 1.9 | .1228 | .75 | 15.764 | 103.1 | –866 |
| 21 | 2.0 | .1582 | –.05 | 21.207 | 2.5 | –1058 |
| 22 | 2.1 | .1182 | –.70 | 16.504 | –91.2 | –741 |
| 23 | 2.2 | .0329 | –.93 | 4.663 | –134.9 | –107 |
| 24 | 2.3 | –.0525 | –.71 | –8.250 | –112.8 | 520 |
| 25 | 2.4 | –.1005 | –.22 | –16.219 | –41.0 | 849 |
| 26 | 2.5 | –.0962 | .29 | –16.060 | 42.6 | 755 |
| 27 | 2.6 | –.0503 | .58 | –8.637 | 98.6 | 324 |
| 28 | 2.7 | .0100 | .57 | 1.945 | 104.0 | –211 |
| 29 | 2.8 | .0558 | .31 | 10.555 | 61.6 | –595 |
| 30 | 2.9 | .0693 | –.04 | 13.468 | –4.3 | –666 |
| 31 | 3.0 | .0501 | –.32 | 9.997 | –61.1 | –426 |

| | | |
|---|---|---|
| $\Sigma(6)^2 = 329048.34$ | $\Sigma(5)^2 = 6660.992256$ | $\Sigma(3)(4) = -0.428921$ |
| $\Sigma(5)(6) = 190.3603$ | $\Sigma(4)(5) = -82.75361$ | $\Sigma(4)(7) = 16887.46$ |
| $\Sigma(4)(6) = 4691.967$ | $\Sigma(3)(5) = +98.2381628$ | $\Sigma(3)^2 = 2.66225444$ |
| $\Sigma(3)(6) = 83.67703$ | $\Sigma(5)(7) = -334446.428$ | $\Sigma(3)(7) = -4835.7037$ |
| $\Sigma(6)(7) = 23599.0$ | $\Sigma(4)^2 = +159.8717$ | |

NACA

TABLE VI.- TAYLOR'S SERIES ITERATION APPLIED TO EXAMPLE III

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Row | $t$ | $\sigma$ | $\sigma'$ | $\rho$ | $\rho'$ | $q_o$ | $\left(\frac{\partial q}{\partial \iota}\right)_o$ | $\left(\frac{\partial q}{\partial \iota'}\right)_o$ | $\left(\frac{\partial q}{\partial \beta}\right)_o$ | $\left(\frac{\partial q}{\partial \beta'}\right)_o$ | $q_m$ | (12)-(7) |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | .1 | .0297 | .0149 | .00191 | .0029 | 3.9370 | -.0110 | -.2114 | .02947 | .00711 | 3.940 | .0030 |
| 3 | .2 | .0692 | .0884 | .00755 | .0138 | 10.9405 | .5410 | -.6242 | .08210 | .04459 | 10.939 | -.0015 |
| 4 | .3 | .0526 | .1813 | .00287 | .0309 | 12.9966 | 1.3609 | -2.3545 | .09799 | .10445 | 12.974 | -.0226 |
| 5 | .4 | .0075 | .222 | -.01278 | .0449 | 5.8985 | -.0912 | -4.3174 | .04535 | .14683 | 5.847 | -.0515 |
| 6 | .5 | .0094 | .226 | -.01141 | .0572 | -7.7700 | -3.0022 | -3.8860 | -.05688 | .13094 | -7.837 | -.0670 |
| 7 | .6 | .0550 | .270 | .01376 | .0850 | -20.3630 | -5.9425 | -1.5001 | -.15168 | .04685 | -20.407 | -.0440 |
| 8 | .7 | .0682 | .364 | .0218 | .1323 | -24.0440 | -8.0511 | 3.5835 | -.18022 | -.07340 | -24.019 | .0250 |
| 9 | .8 | .0270 | .435 | -.0094 | .1757 | -15.6851 | -5.1771 | 9.9342 | -.11866 | -.17151 | -15.576 | .1091 |
| 10 | .9 | .0001 | .447 | -.0317 | .201 | 1.1454 | 2.3728 | 11.6801 | .00683 | -.19527 | 1.300 | .1546 |
| 11 | 1.0 | .0325 | .465 | -.0006 | .235 | 18.0924 | 9.6213 | 7.8174 | .13400 | -.12840 | 18.214 | .1216 |
| 12 | 1.1 | .0700 | .541 | .0385 | .303 | 26.5549 | 14.3646 | -.1106 | .19833 | -.00064 | 26.577 | .0221 |
| 13 | 1.2 | .0500 | .633 | .0150 | .383 | 22.2894 | 12.8773 | -10.8027 | .16760 | .12714 | 22.181 | -.1084 |
| 14 | 1.3 | .0059 | .670 | -.0400 | .436 | 7.5236 | 3.1852 | -17.4231 | .05790 | .19405 | 7.319 | -.2046 |
| 15 | 1.4 | .0113 | .674 | -.0321 | .473 | -10.3501 | -8.5878 | -15.2285 | -.07581 | .16976 | -10.555 | -.2049 |
| 16 | 1.5 | .0573 | .722 | .0345 | .545 | -22.6711 | -17.0053 | -6.4301 | -.16872 | .06866 | -22.775 | -.1039 |
| 17 | 1.6 | .0670 | .817 | .0491 | .658 | -23.8430 | -18.8982 | 6.5720 | -.17860 | -.05885 | -23.785 | .0580 |
| 18 | 1.7 | .0243 | .884 | -.0217 | .753 | -13.9088 | -10.8324 | 18.0802 | -.10522 | -.15221 | -13.711 | .1978 |
| 19 | 1.8 | .0004 | .894 | -.0629 | .805 | 1.8028 | 3.2344 | 20.3838 | .01196 | -.17032 | 2.053 | .2502 |
| 20 | 1.9 | .0353 | .914 | .0020 | .871 | 15.5842 | 15.2862 | 13.2823 | .11543 | -.10984 | 15.764 | .1798 |
| 21 | 2.0 | .0706 | .994 | .0705 | .999 | 21.1848 | 21.0787 | .3580 | .15819 | -.00415 | 21.207 | .0222 |
| 22 | 2.1 | .0474 | 1.084 | .0225 | 1.141 | 16.6536 | 17.1277 | -13.8916 | .12522 | .09490 | 16.504 | -.1496 |
| 23 | 2.2 | .0044 | 1.117 | -.0697 | 1.229 | 4.9112 | 4.1234 | -21.3879 | .03794 | .14266 | 4.663 | -.2482 |
| 24 | 2.3 | .0135 | 1.122 | -.0489 | 1.291 | -8.0209 | -10.1541 | -18.2064 | -.05883 | .12177 | -8.250 | -.2291 |
| 25 | 2.4 | .0594 | 1.174 | .0591 | 1.413 | -16.1158 | -19.3557 | -7.3413 | -.11994 | .04793 | -16.219 | -.1032 |
| 26 | 2.5 | .0656 | 1.269 | .0736 | 1.591 | -16.1305 | -20.0720 | 7.0345 | -.12083 | -.04042 | -16.060 | .0705 |
| 27 | 2.6 | .0216 | 1.333 | -.0386 | 1.734 | -8.9181 | -10.8053 | 18.2719 | -.06694 | -.10187 | -8.637 | .2811 |
| 28 | 2.7 | .0010 | 1.342 | -.0928 | 1.811 | 1.7033 | 3.3439 | 19.9773 | .01174 | -.11129 | 1.945 | .2417 |
| 29 | 2.8 | .0381 | 1.365 | .0096 | 1.913 | 10.3942 | 14.8375 | 12.6147 | .07702 | -.06981 | 10.555 | .1608 |
| 30 | 2.9 | .0710 | 1.447 | .1030 | 2.103 | 13.4568 | 19.4654 | .0845 | .10049 | -.00133 | 13.468 | .0112 |
| 31 | 3.0 | .0447 | 1.534 | .0250 | 2.305 | 10.1343 | 14.9646 | -12.5013 | .07622 | .06024 | 9.997 | -.1373 |

$\Sigma$ (8)$^2$ = 4286.39139342

$\Sigma$ (8)(9) = -228.77281991

$\Sigma$ (8)(10) = 35.627944979

$\Sigma$ (8)(11) = 0.754833829

$\Sigma$ (8)(13) = 1.43276343

$\Sigma$ (9)$^2$ = 4140.34148937

$\Sigma$ (9)(10) = - 4.987891922

$\Sigma$ (9)(11) = -35.532782573

$\Sigma$ (9)(13) = 51.02915144

$\Sigma$ (10)$^2$ = 0.3725473211

$\Sigma$ (10)(11) = 0.0542286301

$\Sigma$ (10)(13) = -0.014427701

$\Sigma$ (11)$^2$ = 0.3777271561

$\Sigma$ (11)(13) = -0.438887876

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TABLE VII.— PRONY'S METHOD APPLIED TO EXAMPLE I

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Row | t | $q_k$ | $q_{k+1}$ | $q_{k+2}$ | 7.02t | cos (6) | sin (6) | 1/2 (3) | 0.91t | $e^{(10)}$ | (9) × (11) |
| 1 | 0.4 | 0.7539 | −1.6600 | −2.9386 | 2.808 | −0.94495 | 0.32722 | 0.37695 | 0.364 | 1.43907 | 0.54246 |
| 2 | .5 | −1.6600 | −2.9386 | −2.7099 | 3.510 | −.93283 | −.36032 | −.83000 | .455 | 1.57617 | −1.30822 |
| 3 | .6 | −2.9386 | −2.7099 | −1.3279 | 4.212 | −.47946 | −.87756 | −1.46930 | .546 | 1.72633 | −2.53650 |
| 4 | .7 | −2.7099 | −1.3279 | .4055 | 4.914 | .20056 | −.97968 | −1.35495 | .637 | 1.89080 | −2.56194 |
| 5 | .8 | −1.3279 | .4055 | 1.6693 | 5.616 | .78586 | −.61841 | −.66395 | .728 | 2.07093 | −1.37499 |
| 6 | .9 | .4055 | 1.6693 | 1.9868 | 6.318 | .99938 | .03525 | .20275 | .819 | 2.26823 | .45988 |
| 7 | 1.0 | 1.6693 | 1.9868 | 1.3773 | 7.020 | .74022 | .67237 | .83465 | .910 | 2.48432 | 2.07354 |
| 8 | 1.1 | 1.9868 | 1.3773 | .2644 | 7.722 | .13105 | .99138 | .99340 | 1.001 | 2.72100 | 2.70304 |
| 9 | 1.2 | 1.3773 | .2644 | −.7775 | 8.424 | −.54024 | .84151 | .68865 | 1.092 | 2.98023 | 2.05234 |
| 10 | 1.3 | .2644 | −.7775 | −1.3026 | 9.126 | −.95590 | .29371 | .13220 | 1.183 | 3.26415 | .43152 |
| 11 | 1.4 | −.7775 | −1.3026 | −1.1666 | 9.828 | −.91955 | −.39298 | −.38875 | 1.274 | 3.57512 | −1.38983 |
| 12 | 1.5 | −1.3026 | −1.1666 | −.5405 | 10.530 | −.44823 | −.89392 | −.65130 | 1.365 | 3.91572 | −2.55031 |
| 13 | 1.6 | −1.1666 | −.5405 | .2181 | 11.232 | .23497 | −.97200 | −.58330 | 1.456 | 4.28877 | −2.50164 |
| 14 | 1.7 | −.5405 | .2181 | .7532 | 11.934 | .80717 | −.59032 | −.27025 | 1.547 | 4.69736 | −1.26946 |
| 15 | 1.8 | .2181 | .7532 | .8672 | 12.636 | .99752 | .07045 | .10905 | 1.638 | 5.14487 | .56105 |

$\Sigma\ (4)^2 = 33.9323$

$\Sigma\ (3)\ (4) = 24.7477$

$\Sigma\ (4)\ (5) = 26.6524$

$\Sigma\ (3)^2 = 33.9334$

$\Sigma\ (3)\ (5) = 6.2232$

$\Sigma\ (7)^2 = 8.1685$

$\Sigma\ (7)\ (8) = -.17988$

$\Sigma\ (7)\ (12) = 2.5253$

$\Sigma\ (8)^2 = 6.8315$

$\Sigma\ (8)\ (12) = 18.2254$

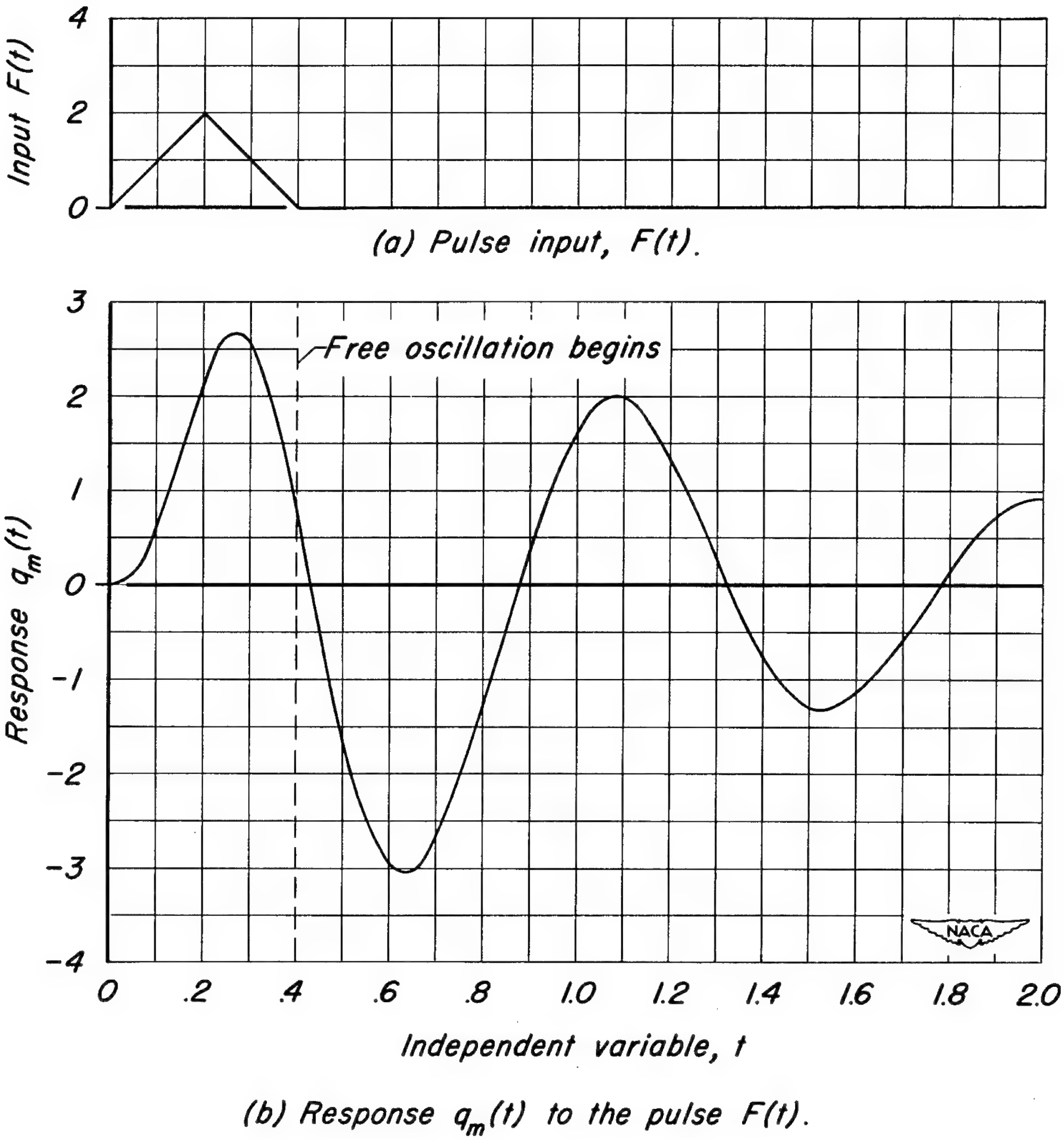
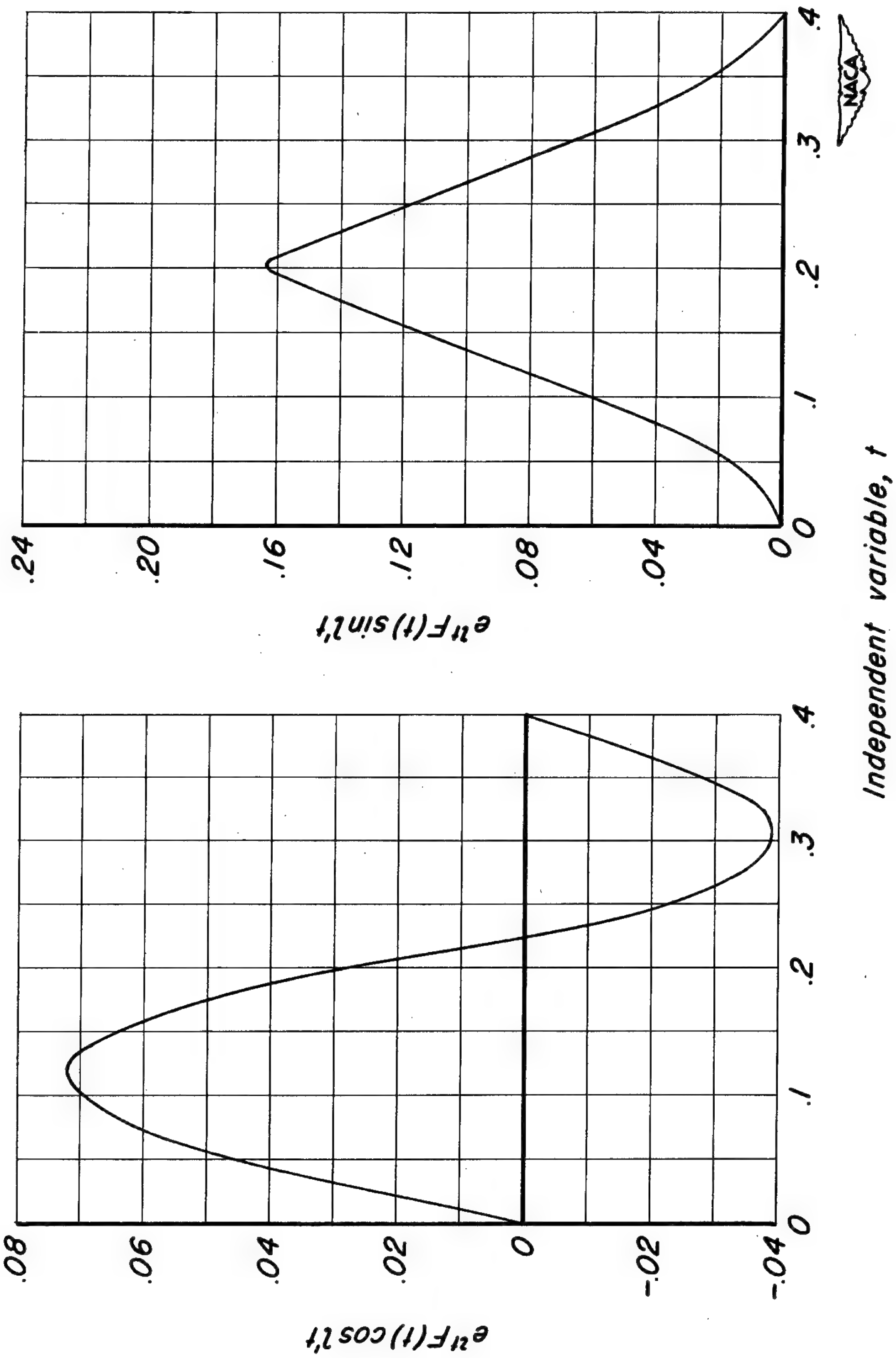



Figure 1.- Pulse input and associated response used in example I.




(a) $e^{lt}F(t)\cos l't$.

(b) $e^{lt}F(t)\sin l't$.

Figure 2.- The variation with time of two quantities used in example I.

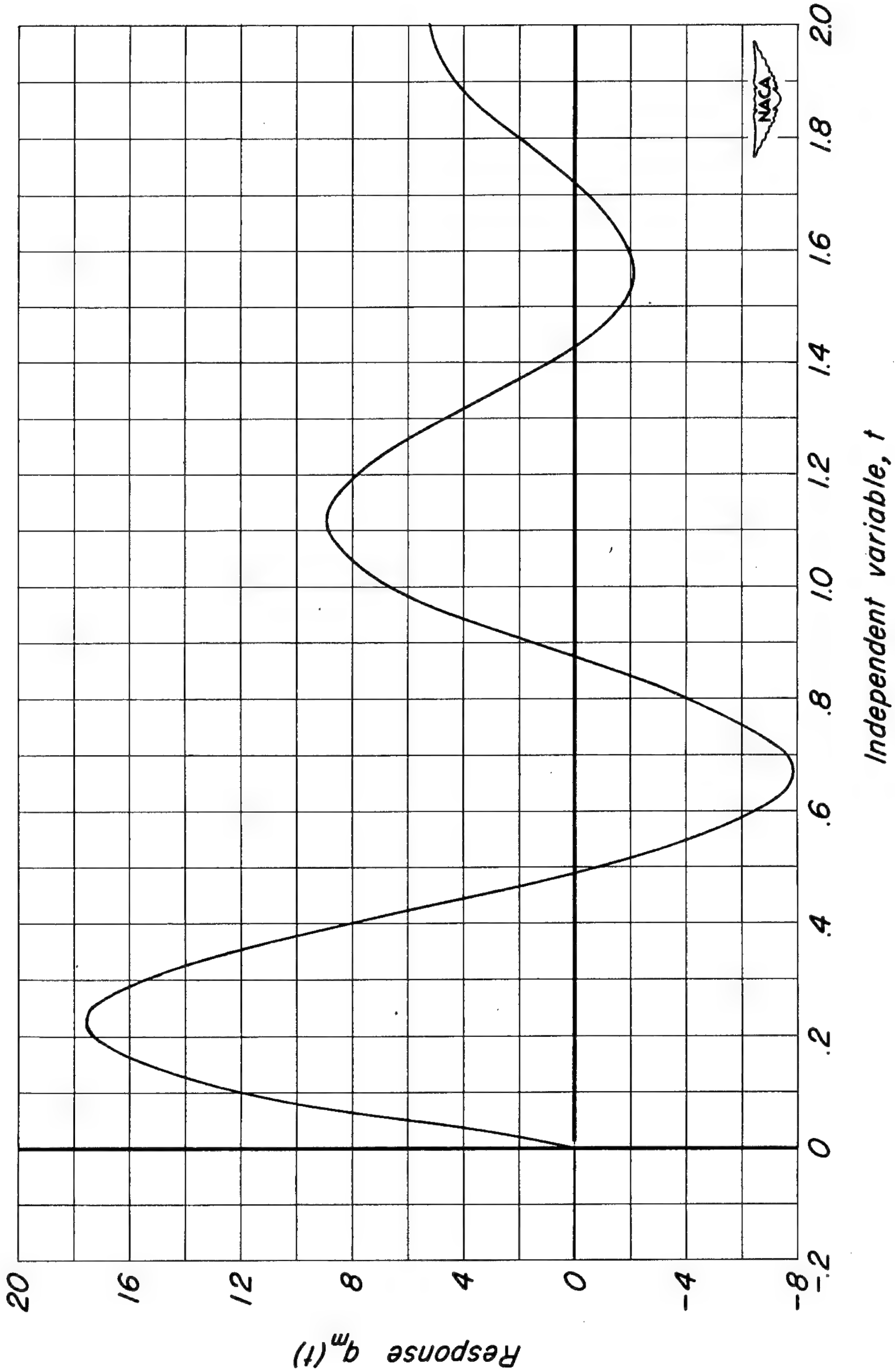
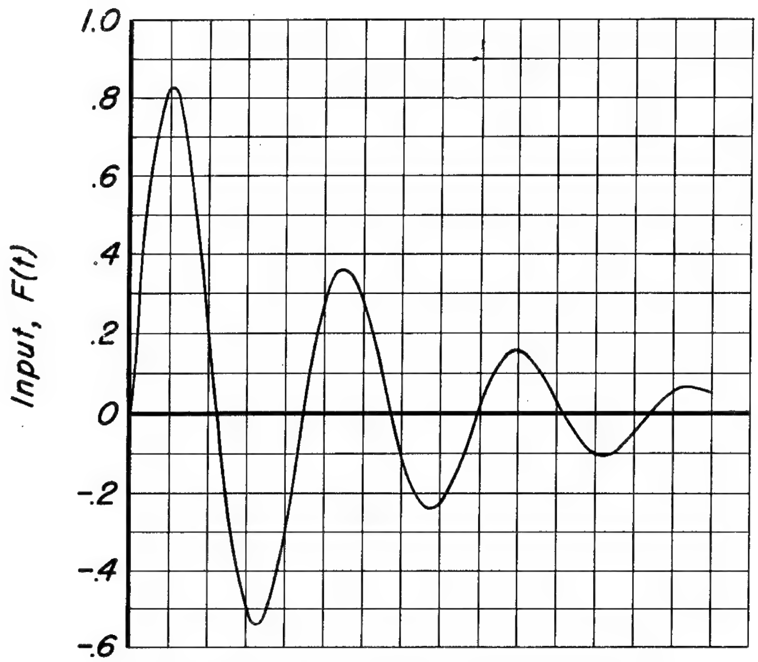
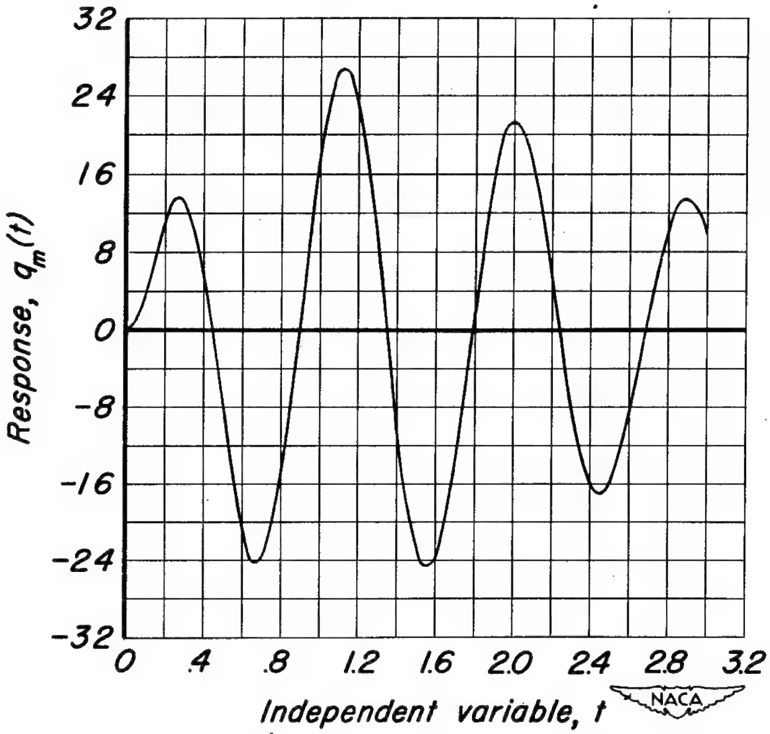



Figure 3.- Response to a step input used in example II.




(a) Input F(t) used in example III.




(b) Response, $q_m(t)$, to input, F(t), of figure 4(a).

Figure 4.- Input and associated response used in example III.

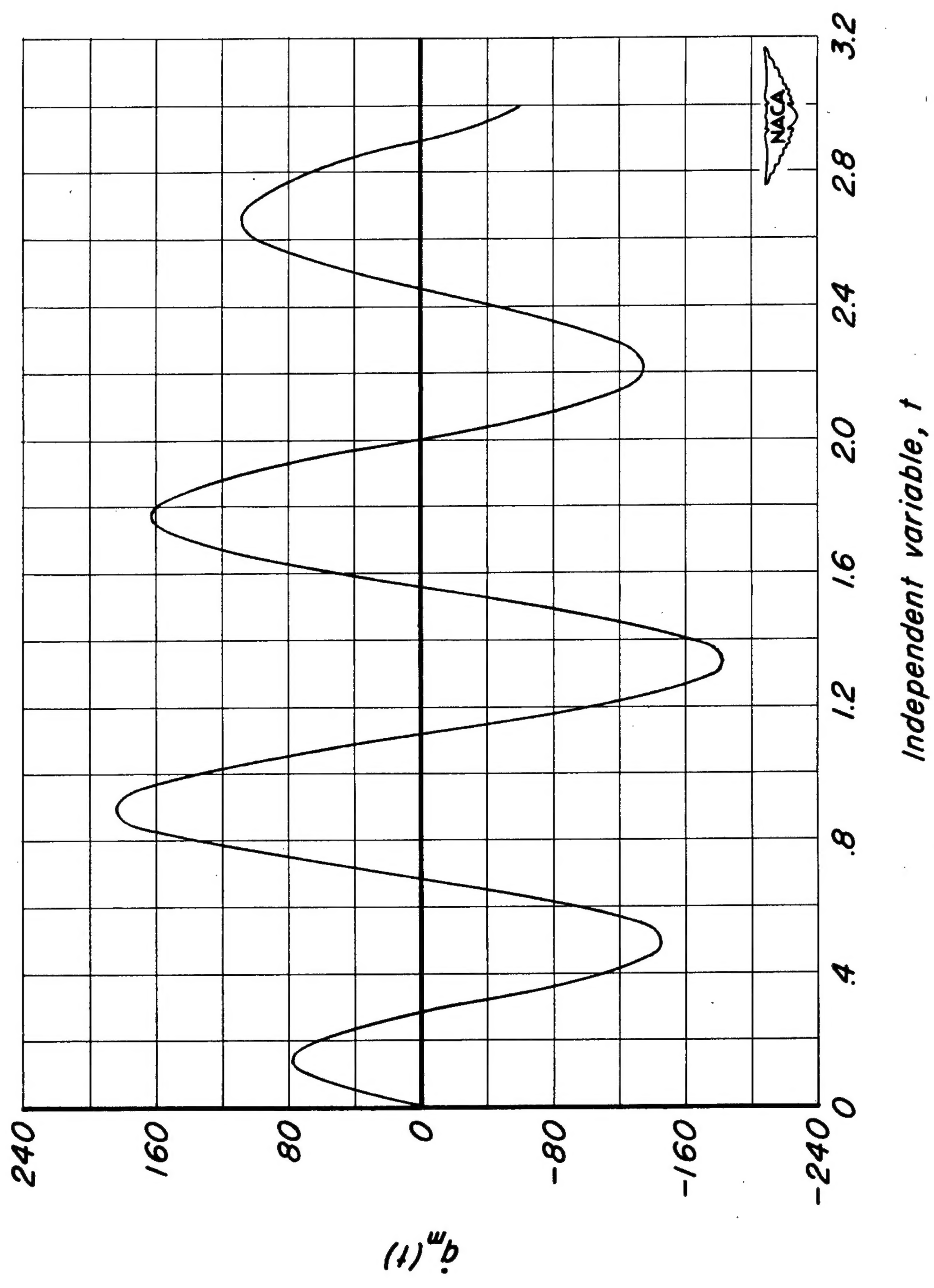



Figure 5.- First derivative, $\dot{q}_m(t)$, of the response, $q_m(t)$, used in example III.

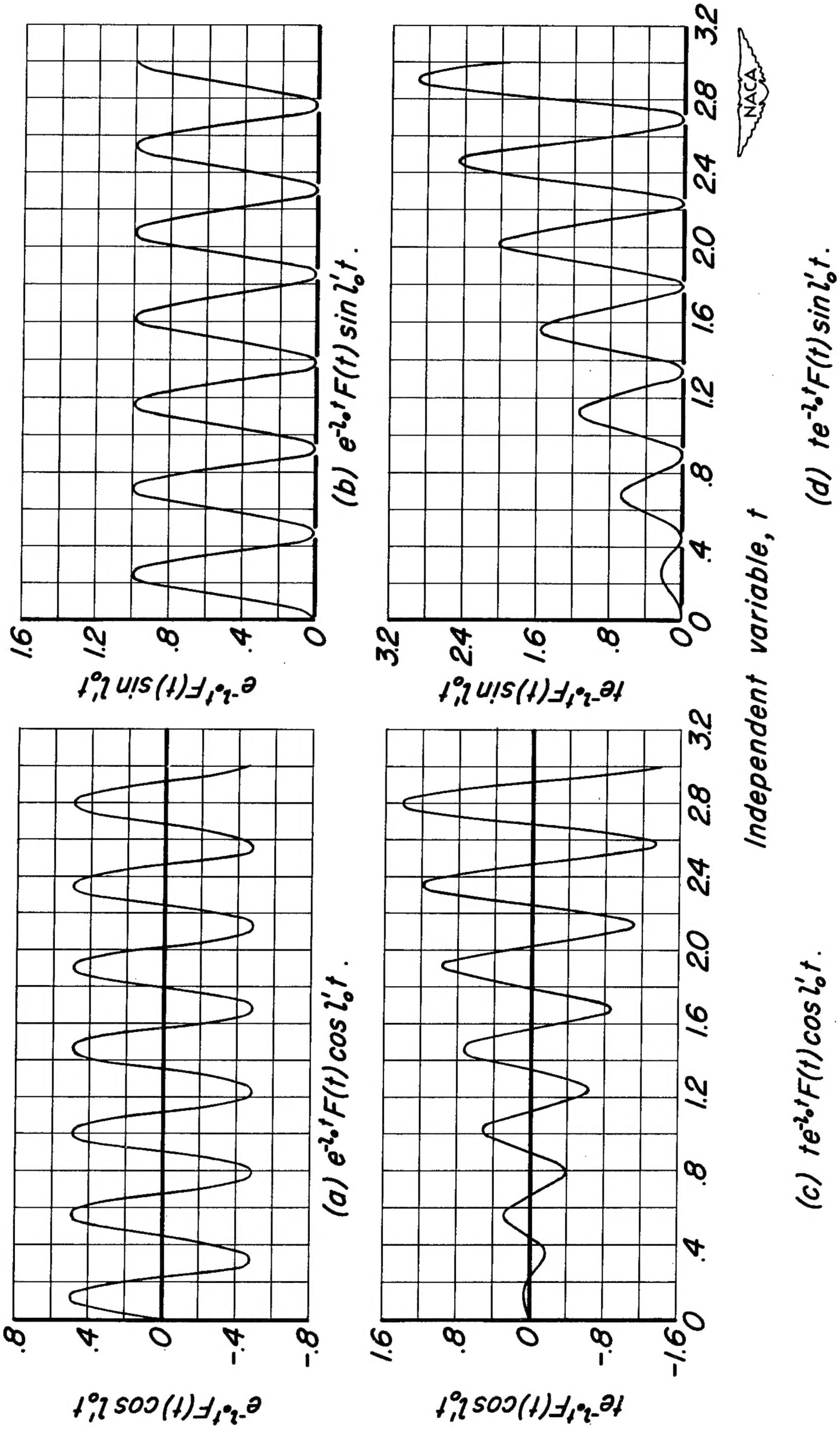



Figure 6.-The variation with time of four quantities used in example III.

## Abstract

A least-squares method for calculating coefficients of a linear differential equation directly from transient-response data is presented. Examples illustrating the application of the method to the calculation of aircraft-stability parameters from the airplane response to an elevator deflection are given.

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Damping Derivatives – Stability 1.8.1.2.3

A Least-Squares Curve-Fitting Method With Applications to the Calculation of Stability Coefficients From Transient-Response Data

By Marvin Shinbrot

NACA TN 2341
April 1951

(Abstract on reverse side)